Software Note

Fit of the Interaction Time Shift in the HERA-B '96 Track Reconstruction

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1 Introduction

This note describes the fit of the interaction time shift for the '96 run track reconstruction. The algorithm can also be applied to single tracks in HERA-B events and may be useful for improved vertex finding strategies. The note presents briefly the fitting technique and the results for a simple detector simulation using the 1996-geometry and assuming a single target wire hit by proton bunches of finite length. The detailed calculations which lead to the fitting algorithm are listed in the appendix for reference.

At the HERA ring the time between bunch crossings is 96 ns. This time structure is used as an input for the read out of the Outer Tracker TDCs. The TDCs deliver the time difference between the HERA bunch clock and the signals in the honeycomb cells. In the ideal case of a pointlike proton bunch this time difference is given by the drift time plus a time independent shift. The latter is due to TDC calibration and cable length variations and differs from channel to channel. For a proton bunch of finite length a second offset of stochastic nature is superimposed: Interactions at the head of the bunch happen earlier and interactions in the bunch tail later than suggested by the HERA bunch clock signal. The time reference t_0 is therefore superimposed by a stochastic offset δt. Its variance σ_t is identical to the r.m.s. bunch length $(c = 1)$[1].

The effect of the t_0 shift is illustrated in Fig. 1. The time measurements in the honeycomb chambers of hits of all tracks from a single pN interaction are shifted in a fully correlated way, deteriorating both the quality of the measured track parameters and the apparent chamber resolution. Therefore, the time shift resembles a stochastic component which correlates not only all coordinates on a track but also all tracks in an event. As a consequence the fitting procedure described in the next section has the task to simultaneously fit all (good) tracks in an event together with the unkown shift δt. Of course, in case of several superimposed interactions in HERA-B, the shift δt can also be extracted for single tracks (or a subset of tracks) by the same method. Since the measured δt values for tracks from the same interaction must be compatible, this gives another handle for vertex recognition. Clearly, the determination of δt can be (and should be) implemented in the vertex package.

2 Fitting Technique

The straight track k is characterized by the parameter vector

$$p^{(k)} = \begin{pmatrix} x_f^{(k)} \\ y_f^{(k)} \\ t_x^{(k)} \\ t_y^{(k)} \end{pmatrix}$$

[1]For the purpose of diagnostics, in order to keep all errors of the fit gaussian, the longitudinal beam profile is assumed to have a gaussian shape. In principle the algorithm is of course also applicable for non-gaussian beam profiles.

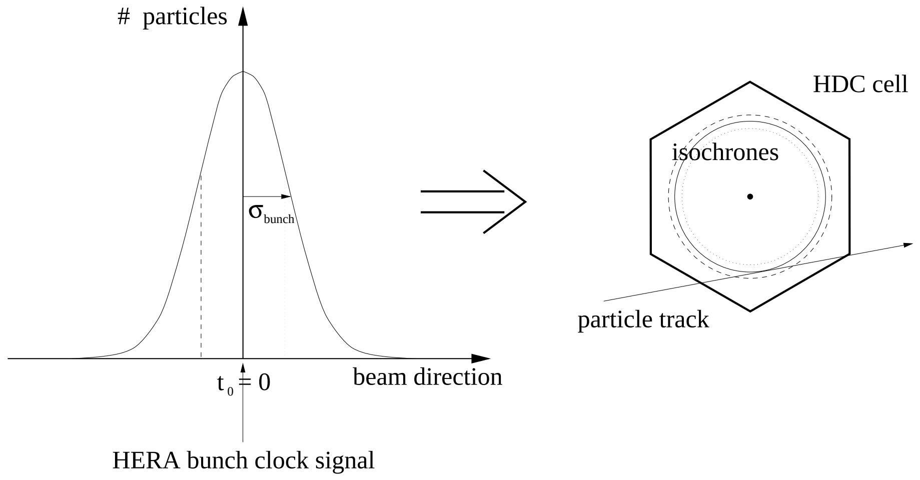

Figure 1: *Influence of the variation of the interaction time shift on the measured drift time.*

where $x_f^{(k)}$ and $y_f^{(k)}$ are the spatial coordinates of the fitted track at a reference z position, and $t_x^{(k)} = \tan\theta_x^{(k)} = (p_x/p_z)^{(k)}$ and $t_y^{(k)} = \tan\theta_y^{(k)} = (p_y/p_z)^{(k)}$ are the corresponding slopes. The coordinate $u_f^{(k)}$ and corresponding slope $\tan\theta_u^{(k)}$ in the measurement direction (perpendicular to the wire direction) is given by

$$\begin{aligned} \tan\theta_u^{(k)} &= t_x^{(k)}\cos\alpha - t_y^{(k)}\sin\alpha \\ u_f^{(k)} &= x_f^{(k)}\cos\alpha - y_f^{(k)}\sin\alpha \end{aligned}$$

where α is the stereo angle of the chamber layer under consideration ($\alpha = +5°, 0°, -5°$). The track index k runs from 1 to N, where N is the number of tracks participating in the fit of the interaction time shift.

The i^{th} coordinate $u_i^{(k)}$ on track k can be decomposed into

- the wire coordinate $u_{W_i}^{(k)}$,
- the u-coordinate relative to the wire,

$$s_i^{(k)}\frac{R_i^{(k)}}{\cos\theta_{u_i}^{(k)}} \quad ,$$

  where $R_i^{(k)}$ is the drift radius in the chamber cell and $s_i^{(k)}$ is the sign specifying whether the track passes on the right or the left of the wire,

- the shift due to δt,

$$-s_i^{(k)}\frac{v_D}{\cos\theta_{u_i}^{(k)}}\delta t$$

  where v_D is the mean drift velocity.

In summary:

$$u_i^{(k)} = u_{W_i}^{(k)} + s_i^{(k)} \frac{R_i^{(k)}}{\cos\theta_{u_i}^{(k)}} - s_i^{(k)} \frac{v_D}{\cos\theta_{u_i}^{(k)}}\,\delta t. \tag{2.1}$$

The problem is slightly non-linear due to the factors $\cos\theta_{u_i}^{(k)}$ which project a drift distance perpendicular to the track into the wire plane. The track angles are, however,

1. well known by initial track fits,
2. very small due to the forward boost of HERA-B events,
3. only a second order correction, since the cosine term deviates from 1 by $\mathcal{O}(\theta_u^2)$.

Therefore, in very good approximation, the cosine terms can be regarded as known without error, converting the problem into an in principle trivial linear χ^2 fit.

The χ^2 function can be written in a compact form, if one uses the vector notation

$$u_0^{(k)} = \begin{pmatrix} \frac{1}{\sigma_1^{(k)}}\left(u_{W_1}^{(k)} + s_1^{(k)}\frac{R_1^{(k)}}{\cos\theta_{u_1}^{(k)}}\right) \\ \vdots \\ \frac{1}{\sigma_i^{(k)}}\left(u_{W_i}^{(k)} + s_i^{(k)}\frac{R_i^{(k)}}{\cos\theta_{u_i}^{(k)}}\right) \\ \vdots \\ \frac{1}{\sigma_{M_k}^{(k)}}\left(u_{W_{M_k}}^{(k)} + s_{M_k}^{(k)}\frac{R_{M_k}^{(k)}}{\cos\theta_{u_{M_k}}^{(k)}}\right) \end{pmatrix} \qquad s^{(k)} = -v_D \cdot \begin{pmatrix} \frac{s_1^{(k)}}{\cos\theta_{u_1}^{(k)}\cdot\sigma_1^{(k)}} \\ \vdots \\ \frac{s_i^{(k)}}{\cos\theta_{u_i}^{(k)}\cdot\sigma_i^{(k)}} \\ \vdots \\ \frac{s_{M_k}^{(k)}}{\cos\theta_{u_{M_k}}^{(k)}\cdot\sigma_{M_k}^{(k)}} \end{pmatrix},$$

where $\sigma_{R_i}^{(k)}$ are the chamber resolutions, $\sigma_i^{(k)} = \sigma_{R_i}^{(k)}/\cos\theta_{u_i}^{(k)}$ their projections into the measurement plane and where M_k is the number of hits on track k. If one uses in addition the geometry matrix

$$\mathbf{A}^{(k)} = \begin{pmatrix} \frac{1}{\sigma_1^{(k)}}\cos\alpha_1 & \cdots & \frac{1}{\sigma_i^{(k)}}\cos\alpha_i & \cdots & \frac{1}{\sigma_{M_k}^{(k)}}\cos\alpha_{M_k} \\ -\frac{1}{\sigma_1^{(k)}}\sin\alpha_1 & \cdots & -\frac{1}{\sigma_i^{(k)}}\sin\alpha_i & \cdots & -\frac{1}{\sigma_{M_k}^{(k)}}\sin\alpha_{M_k} \\ \frac{z_1^{(k)}}{\sigma_1^{(k)}}\cos\alpha_1 & \cdots & \frac{z_i^{(k)}}{\sigma_i^{(k)}}\cos\alpha_i & \cdots & \frac{z_{M_k}^{(k)}}{\sigma_{M_k}}\cos\alpha_{M_k} \\ -\frac{z_1^{(k)}}{\sigma_1^{(k)}}\sin\alpha_1 & \cdots & -\frac{z_i^{(k)}}{\sigma_i^{(k)}}\sin\alpha_i & \cdots & -\frac{z_{M_k}^{(k)}}{\sigma_{M_k}^{(k)}}\sin\alpha_{M_k} \end{pmatrix},$$

where $z_i^{(k)}$ is the z-positions of hit i on track k, the overall χ^2 is given by:

$$\chi^2 = \frac{(\delta t - \delta t_0)^2}{\sigma_{bunch}^2} + \sum_{k=0}^{N}\left[\left(p^{(k)T}\mathbf{A}^{(k)} - u_0^{(k)T} - s^{(k)T}\delta t\right)\left(\mathbf{A}^{(k)T}p^{(k)} - u_0^{(k)} - s^{(k)}\delta t\right)\right]. \tag{2.2}$$

The fit parameters are the unknown time shift δt and the track parameters $p^{(1)}, \cdots, p^{(N)}$. The fit therefore has

$$1 + \sum_k M_k - 1 - 4 \cdot N = \sum_k M_k - 4 \cdot N \quad . \tag{2.3}$$

degrees of freedom. The parameter δt_0 denotes the nominal time shift of the bunch center (normally zero) and σ_{bunch} is the r.m.s. bunch width.

The fit is performed by solving the differential equations

$$\frac{\partial \chi^2}{\partial p^{(k)}} = 0 \tag{2.4}$$

$$\frac{\partial \chi^2}{\partial \delta t} = 0 \quad . \tag{2.5}$$

Since the fit is linear, the solution can be found analytically (see appendix). The resulting fit parameters are

$$\boxed{\begin{array}{rcl} p^{(k)} & = & \mathbf{C}^{(k)}\mathbf{A}^{(k)}\left(u_0^{(k)} + s^{(k)}\delta t\right) \\ \delta t & = & \frac{\sigma_t^2}{\sigma_{bunch}^2}\delta t_0 - \sigma_t^2 \sum_{k=0}^{N} s^{(k)T}\mathbf{B}^{(k)}u_0^{(k)} \end{array}} \tag{2.6}$$

and the relevant errors and covariance matrices read

$$\boxed{\begin{array}{rcl} \mathrm{cov}\left(p^{(k)}\right) & = & \mathbf{C}^{(k)} + \sigma_t^2 \cdot \mathbf{F}^{(k)} \\ \mathrm{cov}\left(\delta t\right) & = & \sigma_t^2 = \frac{1}{K} \end{array}} \tag{2.7}$$

where

$$\boxed{\begin{array}{rcl} \mathbf{C}^{(k)} & = & \left(\mathbf{A}^{(k)}\mathbf{A}^{(k)T}\right)^{-1} \\ \mathbf{F}^{(k)} & = & f^{(k)}f^{(k)T} \\ f^{(k)} & = & \mathbf{C}^{(k)}\mathbf{A}^{(k)}s^{(k)} \\ \mathbf{B}^{(k)} & = & \mathbf{1} - \mathbf{A}^{(k)T}\mathbf{C}^{(k)}\mathbf{A}^{(k)} \\ K & = & \frac{1}{\sigma_{bunch}^2} + \sum_{k=0}^{N} s^{(k)T}\mathbf{B}^{(k)}s^{(k)} \end{array}} \tag{2.8}$$

The matrix $\mathbf{C}^{(k)}$ is at the same time the covariance matrix of the individual track fits ignoring the t_0 shift.

3 Test of the interaction time shift fit

For a test of the interaction time shift fitting procedure, three 1 m high and 17 cm wide honeycomb chamber superlayers, positioned as in the '96 run at $z = 730$ cm, 795 cm and 860 cm relative to the interaction point, were simulated. Each superlayer was assumed to consist of one $+5°$, one $0°$ and one $-5°$ stereo layer (Fig. 2a), each being of the single layer type with ten 5 mm cells and five 10 mm cells per monolayer (Fig. 2b). The bunch was simulated to have a gaussian longitudinal profile with a width of 2 ns, corresponding to $\approx$ 60 cm (Fig. 3). This relatively large value was chosen to test the method even for rather extreme t_0 shifts.

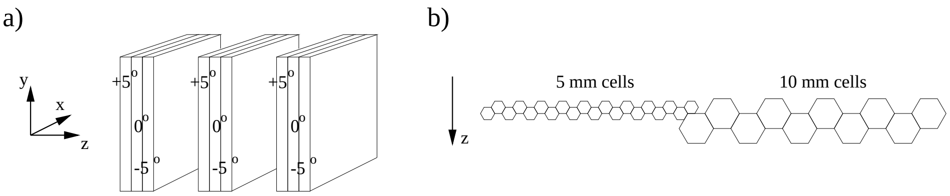

Figure 2: *a) Order of the layers in z direction. b) Cell structure in one layer.*

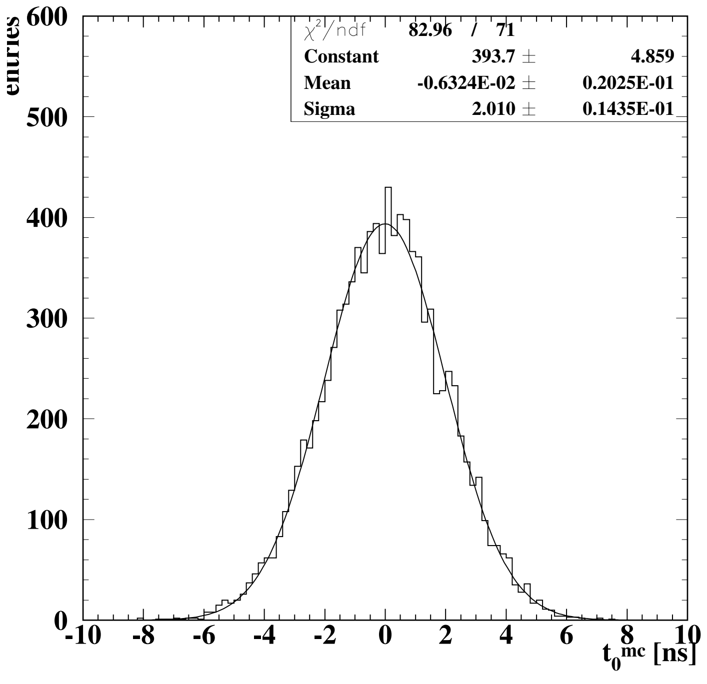

Figure 3: *Distribution of the interaction time t_0 of the simulated events corresponding to a gaussian bunch structure in z.*

In transverse direction, the interaction points were uniformly spread along the target wire. For each event two tracks were generated in the acceptance of the tracking system according to phase space distributions consistent with the expectation from the PYTHIA event generator. The chamber response was simulated with unity hit efficiency and a spatial resolution of $200\,\mu$m. The space drift time relation was assumed to be linear with a drift velocity of $100\,\mu m/ns$. Track deviations from straight lines due to multiple scattering were intentionally ignored. In this way a data sample was produced in which the t_0 algorithm had to perform like an exact χ^2 fit.

The track reconstruction is based on two steps: Firstly, 3–dimensional straight line fits to the measured coordinates were performed using ideal pattern recognition (i.e. using the Monte Carlo information to assign hits to tracks and to resolve the left/right ambiguity). Only tracks with space coordinates in all three superlayers were considered.

This track reconstruction gives a first guess of the track parameters. The badness of the fits is expressed by the pulls[2] of the track parameters (Fig. 4) and the calculated chamber resolution (Fig. 5). The r.m.s. widths of the pull distributions are about 25% larger than expected in case of zero t_0, and the apparent spatial resolution is found to be $\approx 260\,\mu$m which is significantly larger than the $200\,\mu$m put into the drift distance simulation.

The final fit was done with the technique presented in section 2 using the track angles approximated by the initial track fit. Fig. 6, 7 and 8 illustrate the results of the interaction time shift fitting technique:

- The interaction time t_0 is well reconstructed with a resolution of 0.6 ns.
- The pull distribution for the interaction time t_0 is very well compatible with a normal distribution.
- The correlation of the simulated and the reconstructed interaction time t_0 is correct.
- The pull distributions of the track parameters are very well compatible with normal distributions.
- The spatial resolution is close to the value of 200 μm, which was used for the simulation.

In summary, the algorithm passed all systematic tests and seems fully debugged. The t_0 resolution of 0.6 ns achieved with small multiplicity is certainly marginal for separation of interaction vertices, but could be useful for real high multiplicity HERA-B events. The fit of the interaction time shift should hence be by default performed for individual tracks after pattern recognition and initial track fit and probably has to be combined with the final vertex fit.

[2]pull: Difference between real (MC) and reconstructed value divided by the error estimate of the reconstructed value.

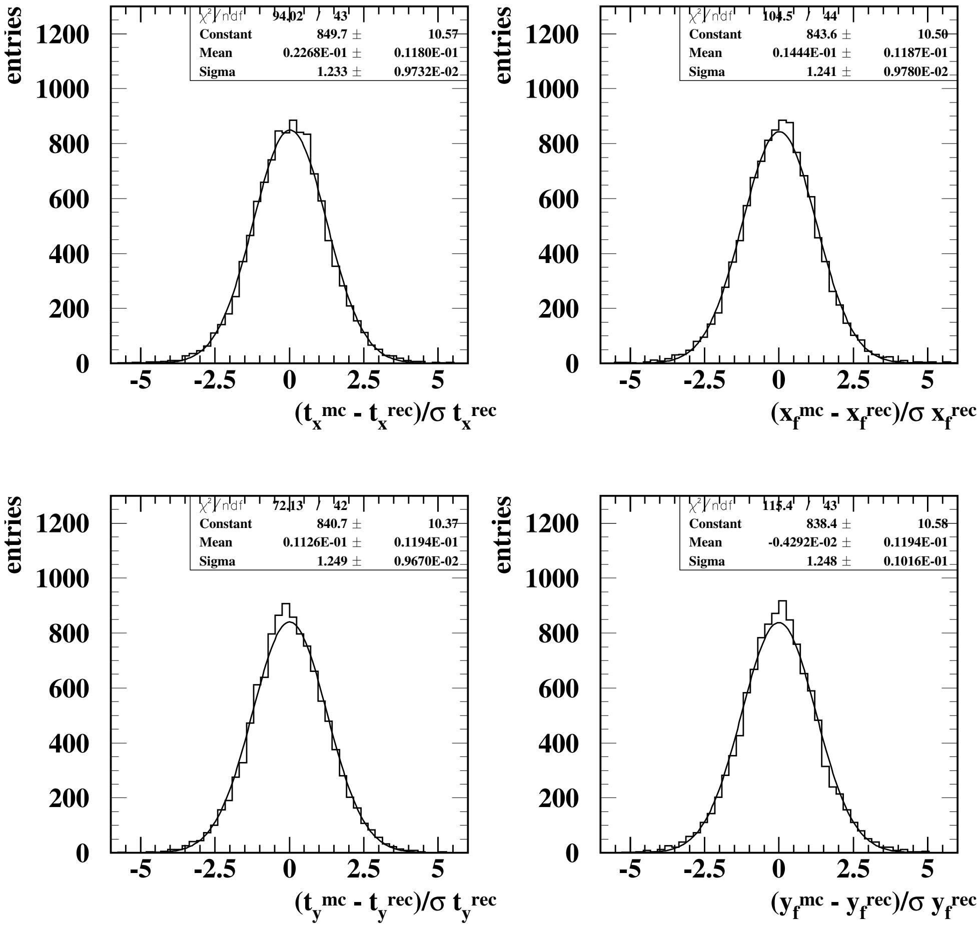

Figure 4: *Pulls of the track parameters before the fit of the interaction time shift.*

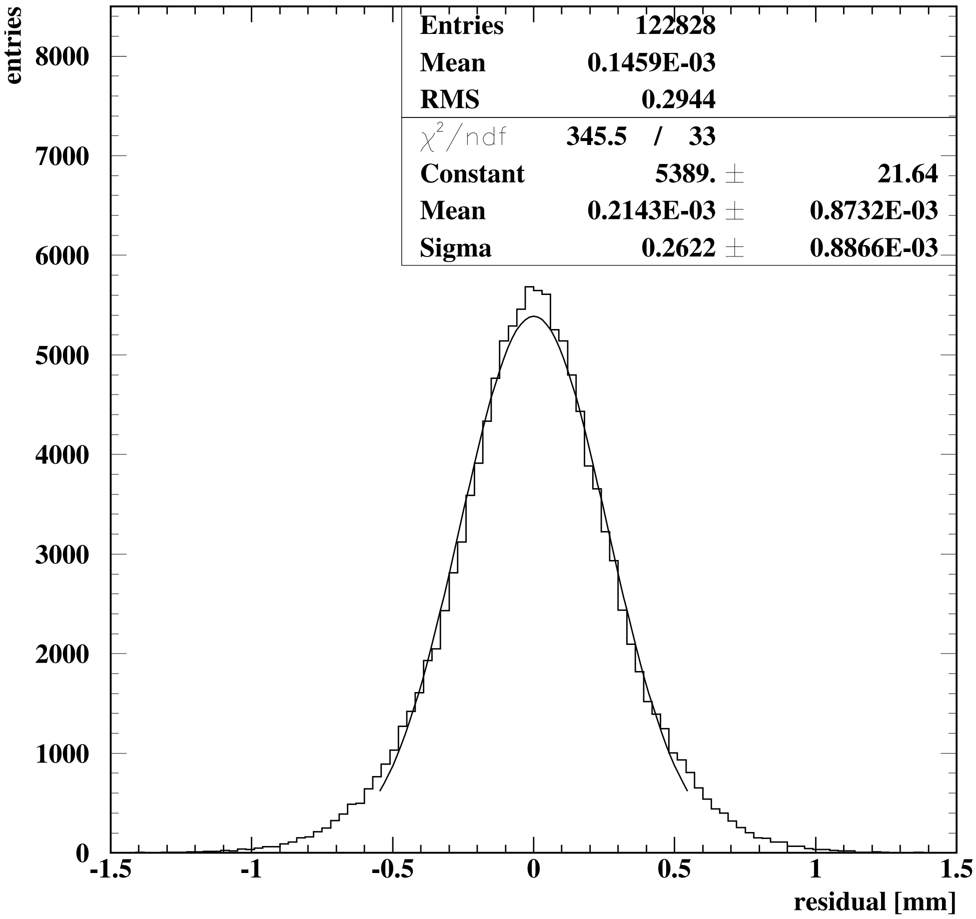

Figure 5: *Distribution of the residuals before the fit of the interaction time shift. The line represents the fit to the data in the range* $\pm 2\sigma$.

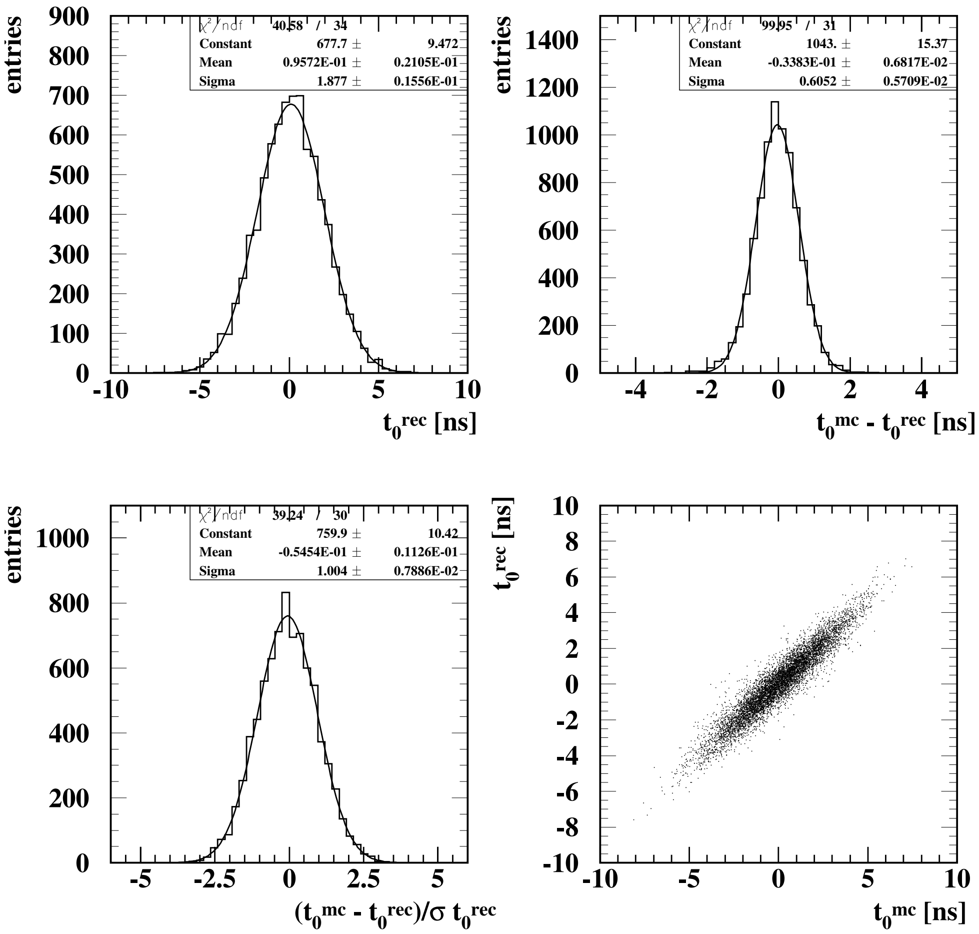

Figure 6: *Distribution of the reconstructed interaction time t_0, of the difference between the simulated (MC) t_0 and the reconstructed t_0 and of the pull for the t_0. The scatter plot shows the correlation between the simulated t_0 and the reconstructed one.*

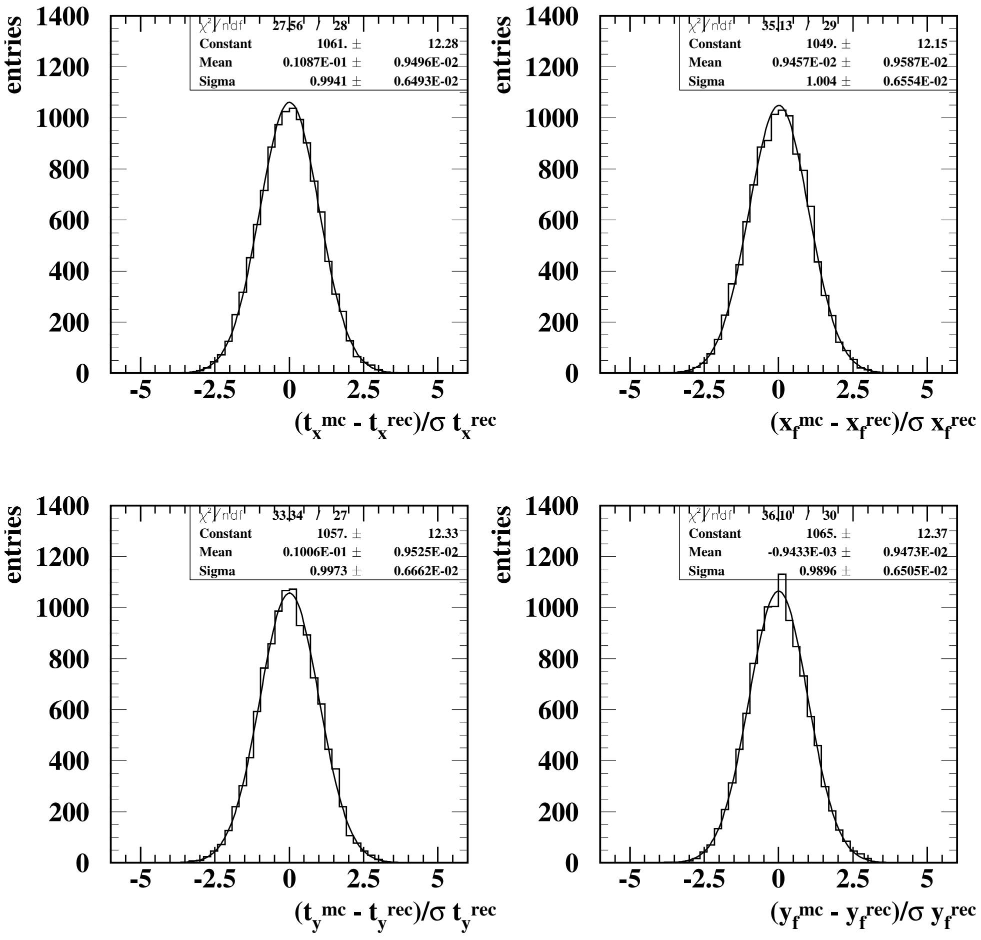

Figure 7: *Pulls of the track parameters after the fit of the interaction time shift.*

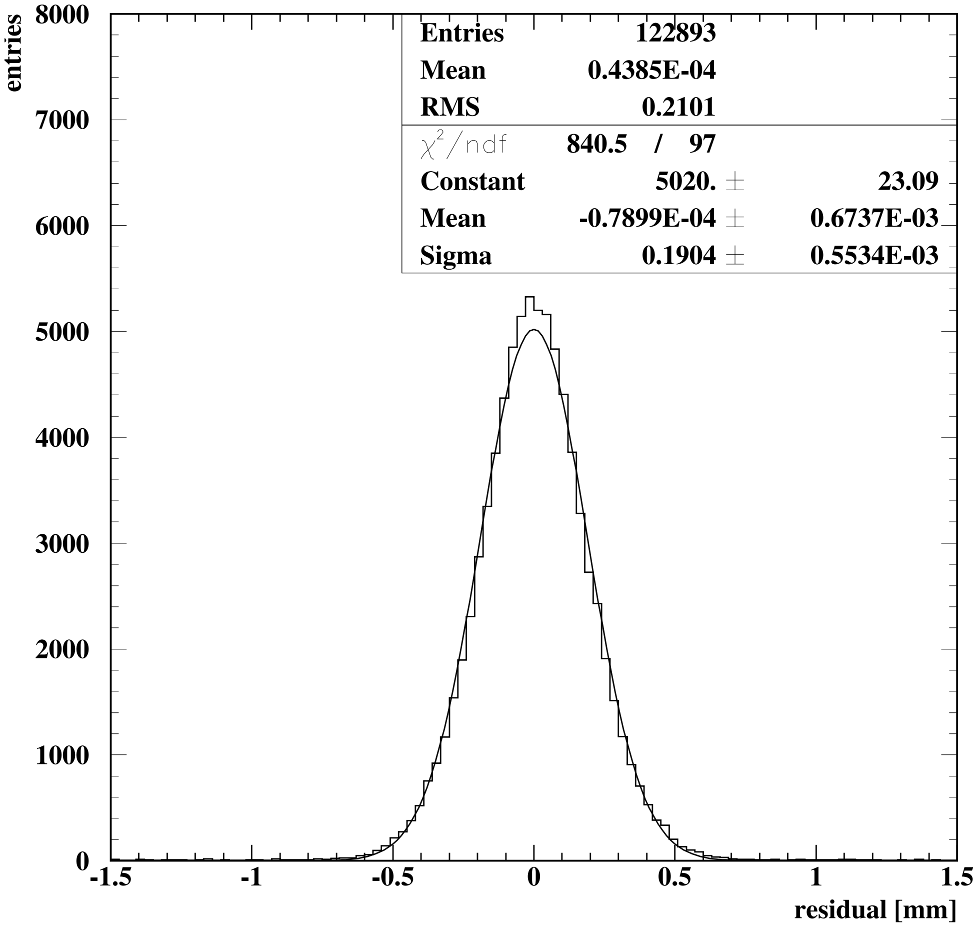

Figure 8: *Distribution of the residuals after the fit of the interaction time shift.*

4 Appendix

Equation 2.4 leads to

$$\frac{\partial \chi^2}{\partial p^{(k)}} = 0$$

$$\begin{aligned} \Rightarrow\ \mathbf{A}^{(k)}\mathbf{A}^{(k)T} p^{(k)} - \mathbf{A}^{(k)}\left(u_0^{(k)} + s^{(k)}\delta t\right) &= 0 \\ \Rightarrow\ p^{(k)} = \mathbf{C}^{(k)}\mathbf{A}^{(k)}\left(u_0^{(k)} + s^{(k)}\delta t\right) & \end{aligned} \tag{4.1}$$

with the matrix

$$\mathbf{C}^{(k)} = \left(\mathbf{A}^{(k)}\mathbf{A}^{(k)T}\right)^{-1}.$$

Equation 2.5 leads to

$$\frac{\partial \chi^2}{\partial \delta t} = 0$$

$$\Rightarrow\ \frac{(\delta t - \delta t_0)^2}{\sigma_{bunch}^2} + \sum_{k=0}^{N}\left(-s^{(k)T}\right)\left(\mathbf{A}^{(k)T}p^{(k)} - u_0^{(k)} - s^{(k)}\delta t\right) = 0$$

$$\Rightarrow\ \delta t\left[\frac{1}{\sigma_{bunch}^2} + \sum_{k=0}^{N} s^{(k)T}s^{(k)}\right] = \frac{\delta t_0}{\sigma_{bunch}^2} + \sum_{k=0}^{N} s^{(k)T}\left(\mathbf{A}^{(k)T}p^{(k)} - u_0^{(k)}\right)$$

$$\begin{aligned} \Rightarrow\ \delta t\left[\frac{1}{\sigma_{bunch}^2} + \sum_{k=0}^{N} s^{(k)T}s^{(k)} - \sum_{k=0}^{N} s^{(k)T}\mathbf{A}^{(k)T}\mathbf{C}^{(k)}\mathbf{A}^{(k)}s^{(k)}\right] &= \\ \frac{\delta t_0}{\sigma_{bunch}^2} + \sum_{k=0}^{N} s^{(k)T}\left(\mathbf{A}^{(k)T}\mathbf{C}^{(k)}\mathbf{A}^{(k)}u_0^{(k)} - u_0^{(k)}\right) & \end{aligned}$$

$$\begin{aligned} \Rightarrow\ \delta t\left[\frac{1}{\sigma_{bunch}^2} + \sum_{k=0}^{N} s^{(k)T}\left(\mathbf{1} - \mathbf{A}^{(k)T}\mathbf{C}^{(k)}\mathbf{A}^{(k)}\right)s^{(k)}\right] &= \\ \frac{\delta t_0}{\sigma_{bunch}^2} + \sum_{k=0}^{N} s^{(k)T}\left(\mathbf{A}^{(k)T}\mathbf{C}^{(k)}\mathbf{A}^{(k)} - \mathbf{1}\right)u_0^{(k)} & \end{aligned}$$

$$\Rightarrow\ \delta t = \frac{\frac{\delta t_0}{\sigma_{bunch}^2} + \sum_{k=0}^{N} s^{(k)T}\left(\mathbf{A}^{(k)T}\mathbf{C}^{(k)}\mathbf{A}^{(k)} - \mathbf{1}\right)u_0^{(k)}}{\frac{1}{\sigma_{bunch}^2} + \sum_{k=0}^{N} s^{(k)T}\left(\mathbf{1} - \mathbf{A}^{(k)T}\mathbf{C}^{(k)}\mathbf{A}^{(k)}\right)s^{(k)}}\quad . \tag{4.2}$$

Together with the definitions

$$\mathbf{B}^{(k)} = \mathbf{1} - \mathbf{A}^{(k)^T} \mathbf{C}^{(k)} \mathbf{A}^{(k)} \tag{4.3}$$

and

$$K = \frac{1}{\sigma^2_{bunch}} + \sum_{k=0}^{N} s^{(k)^T} \mathbf{B}^{(k)} s^{(k)} \tag{4.4}$$

the interaction time shift becomes

$$\delta t = \frac{\frac{\delta t_0}{\sigma^2_{bunch}} - \sum_{k=0}^{N} s^{(k)^T} \mathbf{B}^{(k)} u_0^{(k)}}{K} \quad . \tag{4.5}$$

The error of the interaction time shift reads

$$\sigma_t^2 = \left(\frac{\partial \delta t}{\partial \delta t_0} \right)^2 \cdot \sigma^2_{bunch} + \sum_{k=0}^{N} \sum_{i=0}^{M_k} \sigma^2_{u^{(k)}_{0_i}} \cdot \left(\frac{\partial \delta t}{\partial u^{(k)}_{0_i}} \right)^2 .$$

The relevant partial derivatives of δt are

$$\frac{\partial \delta t}{\partial \delta t_0} = \frac{1}{K \cdot \sigma^2_{bunch}} \tag{4.6}$$

$$\frac{\partial \delta t}{\partial u^{(k)}_{0_i}} = -\frac{1}{K} \cdot \left(s^{(k)^T} \mathbf{B}^{(k)} \right)_i \quad . \tag{4.7}$$

Hence

$$\sum_{i=0}^{N} \sigma^2_{u^{(k)}_{0_i}} \cdot \left(\frac{\partial \delta t}{\partial u^{(k)}_{0_i}} \right)^2 = \frac{1}{K^2} \cdot s^{(k)^T} \mathbf{B}^{(k)} \mathbf{B}^{(k)} s^{(k)} \quad .$$

Using

$$\begin{aligned} \mathbf{B}^{(k)} \mathbf{B}^{(k)} &= \mathbf{1} - 2\mathbf{A}^{(k)^T} \mathbf{C}^{(k)} \mathbf{A}^{(k)} + \mathbf{A}^{(k)^T} \mathbf{C}^{(k)} \mathbf{A}^{(k)} \mathbf{A}^{(k)^T} \mathbf{C}^{(k)} \mathbf{A}^{(k)} \\ &= \mathbf{1} - \mathbf{A}^{(k)^T} \mathbf{C}^{(k)} \mathbf{A}^{(k)} \\ &= \mathbf{B}^{(k)} \end{aligned}$$

one finds

$$\sum_{i=0}^{N} \sigma^2_{u^{(k)}_{0_i}} \cdot \left(\frac{\partial \delta t}{\partial u^{(k)}_{0_i}} \right)^2 = \frac{s^{(k)^T} \mathbf{B}^{(k)} s^{(k)}}{K^2} \quad . \tag{4.8}$$

Using equations 4.6 and 4.8 the error of δt can be rewritten as

$$\sigma_t^2 \;=\; \frac{1}{K^2\sigma_{bunch}^2} + \frac{1}{K^2}\sum_{k=0}^{N} s^{(k)^T}\mathbf{B}^{(k)}s^{(k)} \;=\; \frac{1}{K} \quad . \tag{4.9}$$

The reconstructed δt therefore can be expressed as

$$\delta t = \frac{\sigma_t^2}{\sigma_{bunch}^2}\delta t_0 - \sigma_t^2\sum_{k=0}^{N} s^{(k)^T}\mathbf{B}^{(k)}u_0^{(k)} \quad . \tag{4.10}$$

The covariance matrix for the track parameters can be computed as follows. With equation 4.1 one gets for $p^{(k)}$

$$p^{(k)} = \mathbf{C}^{(k)}\mathbf{A}^{(k)}u_0^{(k)} + \mathbf{C}^{(k)}\mathbf{A}^{(k)}s^{(k)}\frac{\sigma_t^2}{\sigma_{bunch}^2}\delta t_0 - \mathbf{C}^{(k)}\mathbf{A}^{(k)}s^{(k)}\sigma_t^2\sum_{m=0}^{N} s^{(m)^T}\mathbf{B}^{(m)}u_0^{(m)} \quad .$$

The covariance matrix of the track parameters is given by

$$\mathrm{cov}\left(p^{(k)}\right) = \frac{\partial p^{(k)}}{\delta t_0}\frac{\partial p^{(k)^T}}{\delta t_0}\sigma_{bunch}^2 + \sum_{m=0}^{N}\frac{\partial p^{(k)}}{\partial u_0^{(m)}}\frac{\partial p^{(k)^T}}{\partial u_0^{(m)}}\sigma_{u_{0_i}^{(m)}}^2 \quad .$$

The partial derivatives of $p^{(k)}$ are

$$\frac{\partial p^{(k)}}{\partial \delta t_0} = \mathbf{C}^{(k)}\mathbf{A}^{(k)}s^{(k)}\frac{\sigma_t^2}{\sigma_{bunch}^2}$$

$$\begin{aligned}\Rightarrow \frac{\partial p^{(k)}}{\delta t_0}\frac{\partial p^{(k)^T}}{\delta t_0}\sigma_{bunch}^2 &= \frac{\sigma_t^4}{\sigma_{bunch}^2}\cdot \mathbf{C}^{(k)}\mathbf{A}^{(k)}s^{(k)}s^{(k)^T}\mathbf{A}^{(k)^T}\mathbf{C}^{(k)} \\ &= \frac{\sigma_t^4}{\sigma_{bunch}^2}\cdot f^{(k)}f^{(k)^T} \\ &= \frac{\sigma_t^4}{\sigma_{bunch}^2}\cdot \mathbf{F}^{(k)}\end{aligned} \quad ,$$

with

$$\begin{aligned}\mathbf{F}^{(k)} &= f^{(k)}f^{(k)^T} \\ f^{(k)} &= \mathbf{C}^{(k)}\mathbf{A}^{(k)}s^{(k)}\end{aligned}$$

and

$$\frac{\partial p_j^{(k)}}{\partial u_{0_i}^{(m)}} = \left(\mathbf{C}^{(k)}\mathbf{A}^{(k)}\right)_{ji}\cdot\delta_{km} - \left(\mathbf{C}^{(k)}\mathbf{A}^{(k)}s^{(k)}s^{(m)^T}\mathbf{B}^{(m)}\right)_{ji}\cdot\sigma_t^2$$

$$\Rightarrow \frac{\partial p^{(k)}}{\partial u_0^{(m)}} = \mathbf{C}^{(k)}\mathbf{A}^{(k)}\left(\mathbf{1}\delta_{km} - s^{(k)}s^{(m)^T}\mathbf{B}^{(m)}\sigma_t^2\right) \quad .$$

Using 4.3 and $\sigma^2_{u^{(m)}_{0_i}} = 1$ this leads to

$$
\begin{aligned}
\sum_{m=0}^{N} \frac{\partial p^{(k)}}{\partial u_0^{(m)}} \frac{\partial p^{(k)T}}{\partial u_0^{(m)}} &= \mathbf{C}^{(k)}\mathbf{A}^{(k)} \sum_{m=0}^{N} (\mathbf{1}\delta_{km} - \delta_{km}\mathbf{B}^{(k)} s^{(k)} s^{(k)^T} \sigma_t^2 - \delta_{km} s^{(k)} s^{(k)^T} \mathbf{B}^{(k)} \sigma_t^2 + \\
& \quad s^{(k)} s^{(m)^T} \mathbf{B}^{(m)} \mathbf{B}^{(m)} \mathbf{s}^{(m)} s^{(k)^T} \sigma_t^4) \mathbf{A}^{(k)^T} \mathbf{C}^{(k)} \\
&= \mathbf{C}^{(k)} - \sigma_t^2 \mathbf{C}^{(k)} \mathbf{A}^{(k)} \mathbf{B}^{(k)} s^{(k)} s^{(k)^T} \mathbf{A}^{(k)^T} \mathbf{C}^{(k)} - \\
& \quad \sigma_t^2 \mathbf{C}^{(k)} \mathbf{A}^{(k)} s^{(k)} s^{(k)^T} \mathbf{B}^{(k)} \mathbf{A}^{(k)^T} \mathbf{C}^{(k)} + \\
& \quad \mathbf{C}^{(k)} \mathbf{A}^{(k)} s^{(k)} \left(\sum_{m=0}^{N} s^{(m)^T} \mathbf{B}^{(m)} \mathbf{s}^{(m)} \right) s^{(k)^T} \mathbf{A}^{(k)^T} \mathbf{C}^{(k)} \sigma_t^4 \\
&= \mathbf{C}^{(k)} - 2\sigma_t^2 \mathbf{F}^{(k)} + 2\sigma_t^2 \mathbf{C}^{(k)} \mathbf{A}^{(k)} s^{(k)} s^{(k)^T} \mathbf{A}^{(k)^T} \mathbf{C}^{(k)} + \\
& \quad \left(\sum_{m=0}^{N} s^{(m)^T} \mathbf{B}^{(m)} \mathbf{s}^{(m)} \right) \cdot \mathbf{F}^{(k)} \cdot \sigma_t^4 \\
&= \mathbf{C}^{(k)} - \left(\sum_{m=0}^{N} s^{(m)^T} \mathbf{B}^{(m)} \mathbf{s}^{(m)} \right) \cdot \mathbf{F}^{(k)} \cdot \sigma_t^4 \quad .
\end{aligned}
$$

Finally one finds for the covariance matrix of the track parameter

$$
\begin{aligned}
\operatorname{cov}\left(p^{(k)}\right) &= \frac{\sigma_t^4}{\sigma^2_{bunch}} + \mathbf{C}^{(k)} + \left(\sum_{m=0}^{N} s^{(m)^T} \mathbf{B}^{(m)} \mathbf{s}^{(m)} \right) \cdot \sigma_t^4 \cdot \mathbf{F}^{(k)} \\
&= \mathbf{C}^{(k)} + \sigma_t^4 \cdot \mathbf{F}^{(k)} \cdot \left(\frac{1}{\sigma^2_{bunch}} + \sum_{m=0}^{N} s^{(m)^T} \mathbf{B}^{(m)} \mathbf{s}^{(m)} \right) \qquad (4.11) \\
&= \mathbf{C}^{(k)} + \sigma_t^2 \cdot \mathbf{F}^{(k)}.
\end{aligned}
$$