PARTICLE TRACING THROUGH THE HERA-B MAGNETIC FIELD

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Abstract

This note describes particle tracing through inhomogeneous magnetic fields using a fifth order Runge-Kutta method with adaptive step size control. The derivative matrix of the track parameters is provided for the propagation of the track covariance matrix.

1 Introduction

Particle tracing through the inhomogeneous magnetic field is needed for Monte Carlo Simulation in GEANT and for track reconstruction in RANGER. The equation of motion in an inhomogeneous magnetic field cannot be solved analytically. Numerical methods are well studied [1]. The Runge-Kutta method is one of the most common used method and is implemented in GEANT. GEANT uses a fourth order Runge-Kutta method. Fourth order means that the precision depends on the step size to the fifth power. Runge-Kutta methods of any order exist but the fourth order method is the optimal method with respect to CPU time consumption. The precision can be improved by dividing the tracing distance into intervals where each integration is approximated by one Runge-Kutta step. In GEANT this step size is fixed which could cause problems in regions with a large field gradient. Methods with an adaptive step size control adjust the step size to obtain a given numerical precision.

For the track reconstruction a transport routine with automatic step size control has been written. The simplest way to obtain an estimate about the precision is to compare the transport result with two different step sizes. Better is to use a Runge-Kutta method of fifth order where a fourth order Runge-Kutta method is embedded. The difference of the fifth and fourth order solutions is an estimate of the precision. Details about Runge-Kutta methods can be found in [1].

1.1 Equation of motion in a magnetic field

The equation of motion in a magnetic field $\vec{B}$ for a particle with charge e is:

$$\frac{d\vec{P}}{dt} = e\ \vec{v} \times \vec{B} \tag{1}$$

Replacing the time with the trajectory length $dt = ds/|\vec{v}|$ and taking into account that the magnitude of the momentum is constant $|\vec{P}| = P$, on obtains

$$\frac{d\vec{v}}{ds} = \rho \vec{v} \times \vec{B} \tag{2}$$

or

$$\frac{d^2\vec{r}}{ds^2} = \rho \, \frac{d\vec{r}}{ds} \times \vec{B} \tag{3}$$

with $\rho \equiv ec/P$. For the tracking at HERA-B it is necessary to trace particles from one z position to another. Therefore the trajectory length will be replaced by z using

$$\begin{aligned} \frac{d}{ds} &= \frac{dz}{ds}\frac{d}{dz} && (4) \\ \frac{d^2}{ds^2} &= \left(\frac{dz}{ds}\right)^2 \frac{d^2}{dz^2} + \frac{d^2 z}{ds^2}\frac{d}{dz} && (5) \end{aligned}$$

The equation of motion with z as parameter is

$$\begin{aligned} \frac{d^2\vec{r}}{dz^2} &= \left(\frac{dz}{ds}\right)^{-2} \left[\rho \frac{dz}{ds}\,\frac{d\vec{r}}{dz} \times \vec{B} - \frac{d^2 z}{ds^2}\frac{d\vec{r}}{dz}\right] && (6) \\ &= \rho \left(\frac{dz}{ds}\right)^{-1} \left[\frac{d\vec{r}}{dz} \times \vec{B} - \frac{d\vec{r}}{dz}\left(\frac{d\vec{r}}{dz} \times \vec{B}\right)_3\right] && (7) \\ &= \tilde{\rho}\sqrt{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2} \left[\frac{d\vec{r}}{dz} \times \vec{B} - \frac{d\vec{r}}{dz}\left(\frac{d\vec{r}}{dz} \times \vec{B}\right)_3\right] && (8) \end{aligned}$$

with $\tilde{\rho} \equiv \rho \cdot sign\left(\frac{dz}{ds}\right)$. This is equivalent to a system of first order differential equations:

$$\frac{d}{dz}\begin{pmatrix} z \\ x \\ y \\ x' \\ y' \\ \tilde{\rho} \end{pmatrix} = \begin{pmatrix} 1 \\ x' \\ y' \\ \tilde{\rho}\sqrt{1 + x'^{\,2} + y'^{\,2}}\left[(\vec{r}' \times \vec{B})_1 - x'(\vec{r}' \times \vec{B})_3\right] \\ \tilde{\rho}\sqrt{1 + x'^{\,2} + y'^{\,2}}\left[(\vec{r}' \times \vec{B})_2 - y'(\vec{r}' \times \vec{B})_3\right] \\ 0 \end{pmatrix} \tag{9}$$

Defining $\vec{p} \equiv (z, x, y, x', y', \tilde{\rho})$ we have

$$\frac{d}{dz}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix} = \begin{pmatrix} 1 \\ p_4 \\ p_5 \\ p_6\sqrt{1 + p_4^2 + p_5^2}\,[p_4 p_5 B_1 - (1 + p_4^2)B_2 + p_5 B_3] \\ p_6\sqrt{1 + p_4^2 + p_5^2}\,[(1 + p_5^2)B_1 - p_4 p_5 B_2 - p_4 B_3] \\ 0 \end{pmatrix} \tag{10}$$

or in vector notation

$$\frac{d}{dz}\vec{p} = \vec{f}(\vec{p}) \tag{11}$$

i	b_{ij}					c_i	c_i^*
1						$\frac{37}{378}$	$\frac{2825}{27648}$
2	$\frac{1}{5}$					0	0
3	$\frac{3}{40}$	$\frac{9}{40}$				$\frac{250}{621}$	$\frac{18575}{48384}$
4	$\frac{3}{10}$	$-\frac{9}{10}$	$\frac{6}{5}$			$\frac{125}{594}$	$\frac{13525}{55296}$
5	$-\frac{11}{54}$	$\frac{5}{2}$	$-\frac{70}{27}$	$\frac{35}{27}$		0	$\frac{277}{14336}$
6	$\frac{1631}{55296}$	$\frac{175}{512}$	$\frac{575}{13824}$	$\frac{44275}{110592}$	$\frac{253}{4096}$	$\frac{512}{1771}$	$\frac{1}{4}$
$j =$	1	2	3	4	5		

Table 1: Cash-Karp parameters for embedded Runge-Kutta method.

2 Numerical integration of the equation of motion

2.1 Transport of track parameters

The problem of particle tracing is an initial value problem of the equation of motion. The track parameters at a position $z + h$ has to been determined from the given parameters $\vec{p}_0$ at position z. Within the fifth order Runge-Kutta method [1] the new parameters are:

$$\vec{p}(h) = \vec{p}_0 + \sum_{m=1}^{6} c_m \vec{k}^m \tag{12}$$

where $\vec{k}^m$ is an estimate for $\Delta\vec{p}$ obtained from the derivatives $\vec{f}$ at the position $\vec{p}^{\,m}$:

$$\vec{k}^m = h\ \vec{f}(\vec{p}^{\,m}) \tag{13}$$

$$\vec{p}^{\,m} = \vec{p}_0 + \sum_{n=1}^{m-1} b_{mn}\ \vec{k}^n \tag{14}$$

The values of c_m and b_{mn} (see table 1) are chosen to get a precision proportional to h^6. Replacing c_m with c_m^* in equation 12 gives a precision proportional to h^5 (fourth order Runge-Kutta).

The estimated error is

$$\vec{\Delta} = \sum_{m=1}^{6} (c_m - c_m^*)\vec{k}^m \propto h^5 \tag{15}$$

If this is larger that the required error $\vec{\Delta}_r$ the step has to be redone. In any case an optimized new step size can be derived. Since the error is proportional to h^5 the optimal step size is given by:

$$h_n = S h \min_i \left| \frac{\Delta_r^i}{\Delta^i} \right|^{0.25} \tag{16}$$

This already takes into account an assumed error accumulation proportional to h for all steps ($\Delta_r^i \propto h_n$). S is a safety factor a few percent below 1. The new step size is calculated for each step and is used for the next step.

2.2 Propagation of the error matrix

The covariance matrix of the track parameters $\vec{p}$ propagates from z_0 to z according to

$$V_{ij}(z) = \sum_{m,n} \frac{\partial \vec{p}_i}{\partial \vec{p}_m^{\,0}} V_{mn}(z_0) \frac{\partial \vec{p}_j}{\partial \vec{p}_n^{\,0}} \tag{17}$$

From equation 12 and $h = z - z_0$ follows:

$$\frac{\partial \vec{p}_i}{\partial \vec{p}_j^{\,0}} = \delta_{ij} + \sum_m c_m \frac{\partial \vec{k}_i^m}{\partial \vec{p}_j^{\,0}} \tag{18}$$

$$\frac{\partial \vec{k}_i^m}{\partial \vec{p}_j^{\,0}} = h \sum_l \left.\frac{\partial \vec{f}_i}{\partial \vec{p}_l}\right|_{\vec{p}^{\,m}} \left(\frac{\partial \vec{p}_l^{\,0}}{\partial \vec{p}_j^{\,0}} + \sum_{n=1}^{m-1} b_{mn} \frac{\partial \vec{k}_l^n}{\partial \vec{p}_j^{\,0}} \right) - \frac{\partial z_0}{\partial \vec{p}_j^{\,0}} \vec{f}_i(\vec{p}^{\,m}) \tag{19}$$

For an inhomogeneous magnetic field the field strength $\vec{B}$ is a function of p_1, p_2, p_3 and derivatives of $\vec{B}$ enter in the equations above. However the effect is small and will be neglected in the numerical solution.

3 Program code

The track tracing is done by four subroutines:

`RKMAIN` : Main routine, calls `RKSTEP, RKNEWH, RKDERI`. This routine has to be called for the tracing.
`RKSTEP` : One Runge-Kutta step, calculates the track parameters at a new z position (equation 12)
`RKNEWH` : calculates the new step size (equation 16)
`RKDERI` : calculates the derivatives (equation 18)

Figure 1 gives an overview of the program code. The routine `RKMAIN` first performs one Runge-Kutta integration with the input step size `FSTEP`. If this step size turns out to be too large to meet the required precision according to equation 16, the Runge-Kutta step is redone with the new step size. Otherwise the derivatives (equation 18) are calculated. This procedure repeats (with the new step size) until the final z position `ZOUT` is reached.

3.1 RKMAIN

```
CALL RKMAIN( ICOVAR, ZOUT, FSTEP, PIN, PREC, POUT, RKD,
             IERROR)
```

Input:

`ICOVAR`	I	this variable will be included in new ARTE versions to chose no, normal or fast covariance matrix calculation
`ZOUT`	R*8	z position of new track parameters
`FSTEP`	R*8	initial Runge-Kutta step size
`PIN(6)`	R*8	track parameters $(z, x, y, tx, ty, Q/p)$ as in `RTRA` bank
`PREC(4)`	R*8	tracing precision for (x,y,tx,ty) per cm distance in z

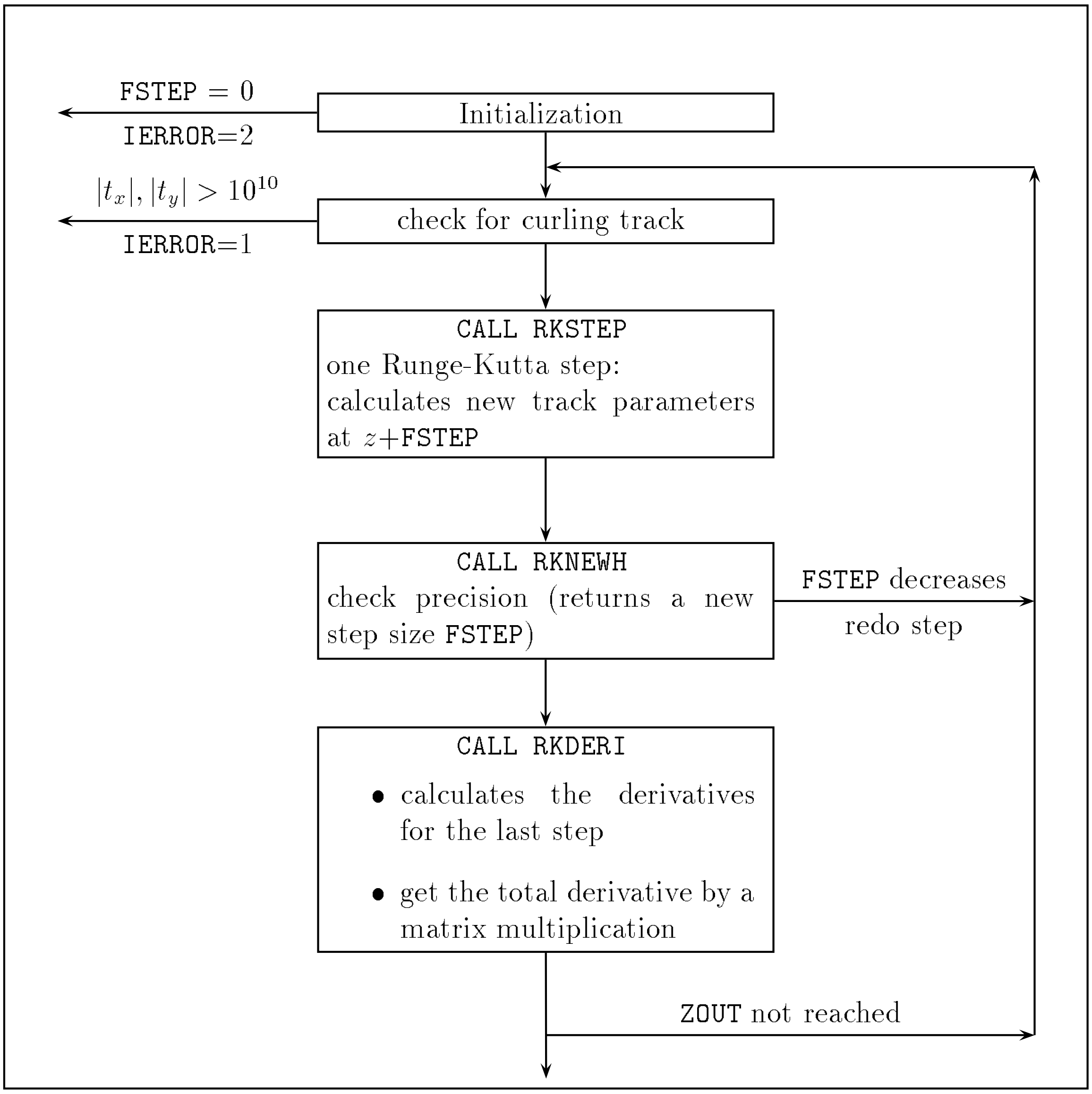

Figure 1: Main transport routine RKMAIN.

Output:

`FSTEP`	R*8	estimated step size to meet the tracing precision
`POUT(6)`	R*8	track parameters $(z, x, y, tx, ty, Q/p)$ at $z =$`ZOUT`
`RKD(5,5)`	R*8	derivatives $\dfrac{d\texttt{POUT}(2-6)}{d\texttt{PIN}(2-6)}$
`IERROR`	I	$= 0$ ok
		$= 1$ curling track
		$= 2$ input `FSTEP` $= 0$ or `ZOUT` $=$ `PIN(1)`

This is the main tracing routine which has to been called by the user. The tracing works in both z directions, `ZOUT` might be larger or less than `PIN(1)`. An initial Runge-Kutta step size of `FSTEP` = `ZOUT` - `PIN(1)` is recommended. The `PREC` array specifies the uncertainty which can be tolerated. For pattern recognition it should be a factor of 10 smaller than the hit resolution. It has to been given as precision per tracking distance. If a precision of $\Delta x = 0.001$ cm for a tracing from $z_0 = 200$ cm to $z_1 = 100$ cm is required, `PREC(1)` has to be set to $\frac{0.001}{|200-100|} = 10^{-5}$. For details how to chose `FSTEP` and `PREC` see the next section. The covariance matrix propagates according to equation 17 using `RKD(5,5)`.

Inside the routine a minimal step size `HMIN` is defined. By changing this value a maximum computation time per tracing length can be forced.

3.2 RKSTEP

```
CALL RKSTEP(  H, PIN, FAC, POUT, DELTA )
```

Input:

`H`	R*8	step size (h in equation 13)
`PIN(*)`	R*8	track parameters $(z, x, y, tx, ty, \tilde{\rho})$
`FAC`	R*8	$\tilde{\rho} = Qc/p$ [cm^{-1} kG^{-1}]

Output:

`POUT(*)`	R*8	track parameters at $z + h$
`DELTA(4)`	R*8	estimated error (equation 15) for (x, y, tx, ty)

`RKSTEP` is called by `RKMAIN` to performe one Runge-Kutta step without calculating the derivative matrix. The derivative matrix is only calculated if the step doesn't have to be redone.

3.3 RKNEWH

```
CALL RKNEWH(  H, DELTA, PREC, OK, HNEW )
```

Input:

`H`	R*8	last step size
`DELTA(4)`	R*8	output from `RKSTEP`
`PREC(4)`	R*8	input to `RKMAIN`

Output:

`OK`	B	= `.TRUE.` precision reached
`HNEW`	R*8	new estimate for step size

RKNEWH determines a new step size according to equation 16 from DELTA and PREC. OK is set to .TRUE. if $\min_i \left|\frac{\Delta^i_r}{\Delta^i}\right| \geq 1$ (see equation 16).

3.4 RKDERI

```
CALL RKDERI( UPDATE, H, EC, RKH )
```

Input:

UPDATE	B	=.TRUE. update of derivatives (necessary for more than one Runge-Kutta step per call of RKMAIN)
H	R*8	step size
EC	R*8	$c\ [\frac{\text{GeV}}{\text{e kG}}\text{cm}^{-1}]$

Output:

RKH(5,5)	R*8	derivatives $\frac{d\texttt{POUT}(2-6)}{d\texttt{PIN}(2-6)}$

This routine calculates the derivatives. It is called from RKMAIN. If more than one Runge-Kutta step are necessary to trace to ZOUT the total derivative matrix is obtained from a matrix multiplication:

$$\frac{d\vec{p}^{\;n}}{d\vec{p}^{\;0}} = \sum_{i,j,..} \frac{d\vec{p}^{\;n}}{d\vec{p}_i^{\;n-1}} \frac{d\vec{p}_i^{\;n-1}}{d\vec{p}_j^{\;n-2}} \cdots \frac{d\vec{p}^{\;1}}{d\vec{p}^{\;0}}$$

The present version of ARTE (ARTE-01-04-r3, ARTE-01-05-r4) contains the original RKDERI routine. This routine has largely been improved by T. Kuhn-Sander reducing the time consumption by a factor of two. The timing in table 2 was determined with the new routine.

4 Performance

4.1 Time Consumption

The time consumption for one Runge-Kutta step has been measured (table 2). 80 % of the time is needed for the calculation of the derivative matrix. The total time for one Runge-Kutta step with calculation of the derivative matrix is about 300 μsec on a R10000 CPU (hera-b).

4.2 Precision and step size

The relation between precision and step size has been studied. The field strength and gradient dB_y/dz for the region of interest is shown in figure 2, 3.

The step size as a function of the required precision for tracing distances of 50 and 100 cm, different particle momenta and z positions is shown in figure 4 - 11. The plots show the value of FSTEP which is returned when tracing +20 cm starting at the z position indicated above the figure for a given particle momentum and precision. The precision is varied for one component (x, y, t_x or t_y) while setting the precision of the other components to a large value.

parameter evaluation	47 μsec
calculation of magnetic field (UTFELD)	17 μsec
calculation of derivatives	253 μsec
step size control	$\approx 0\mu$sec
total for one step+derivatives	317 μsec

Table 2: Time consumption on `hera-b` (R10000 CPU, 194 MHZ, IRIX 6.2) for one Runge-Kutta step.

For moderate precisions a step size of 100 cm is sufficient. Therefore in most of the cases the tracing from one detector layer to the next needs only one Runge-Kutta step. The step size for 50 cm (figure 4 - 7) and 100 cm tracing distance (figure 8 - 11) show a small deviation as expected from the relation $\Delta \propto h^5$.

To check whether the step size evaluation guarantees the required precision, Runge-Kutta tracing was done for all points in figures 4 - 11 with a tracing distance equal to `FSTEP`. The tracing was redone with a much higher precision and the two results were compared. The ratio of the difference over the required precision shows that the evaluated step size usually gives the required precision (figure 12). For large step sizes there is a tendency for an overestimation of the step size (figure 13).

4.3 Conclusions

The following conclusions can be drawn from the studies:

- The step size adaption works properly for most cases. At large tracing distances the evaluated step size tends to be too large.
- For most purposes a step size of $\approx$ 100 cm is sufficient.
- Since the necessary step size is of the same order of magnitude as the normal tracing distance the chosen method seems to be adequate. Methods which are more efficient for high precision requirements [1] can most probably not improve the performance.
- In rare cases the required precision can not be reached. The step size is then taken to be `HMIN` (see routine `RKMAIN`).
- Most of the time is spend by calculating the derivatives.

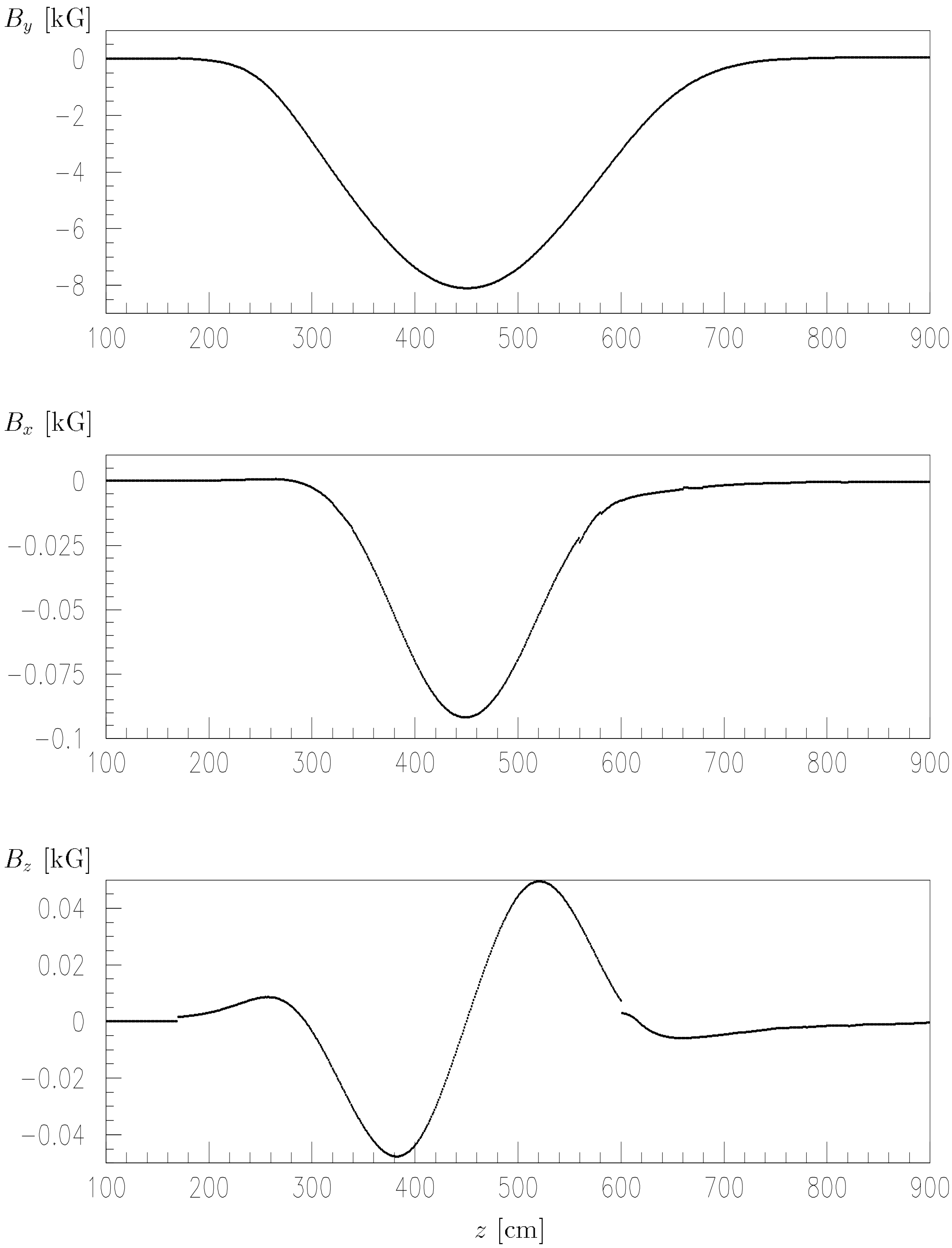

Figure 2: y, x and z component of the magnetic field as a function of z at $x = y = 0$.

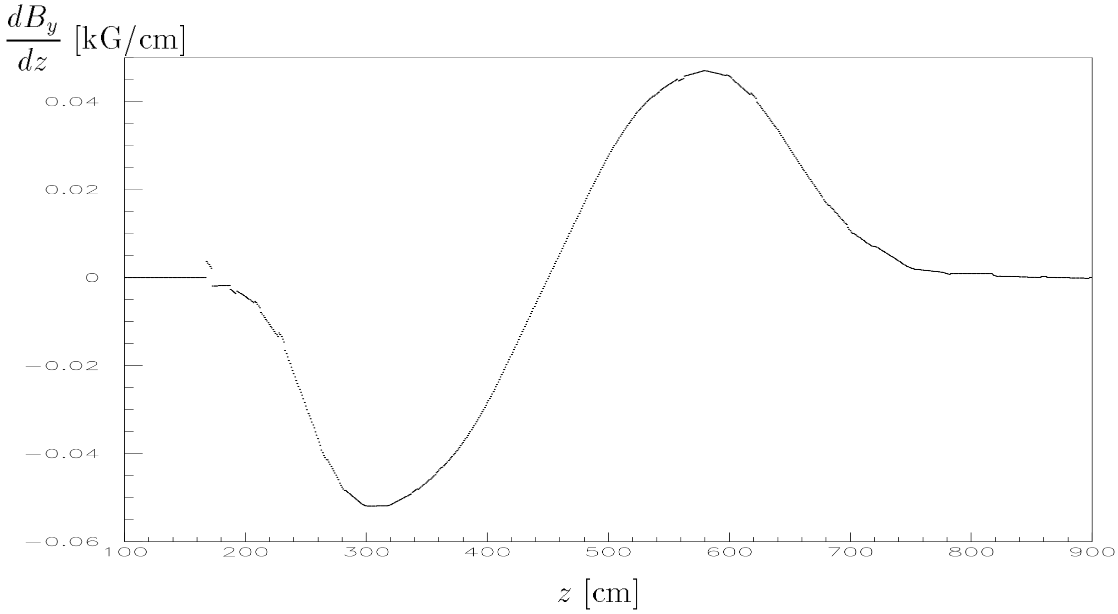

Figure 3: Magnetic field gradient dB_y/dz as a function of z at $x = y = 0$.

References

[1] *Numerical Recipes in C*; W. Press, S. Teukolsky, W. Vetterling, B. Flannery, Cambridge University Press, second edition (1995).

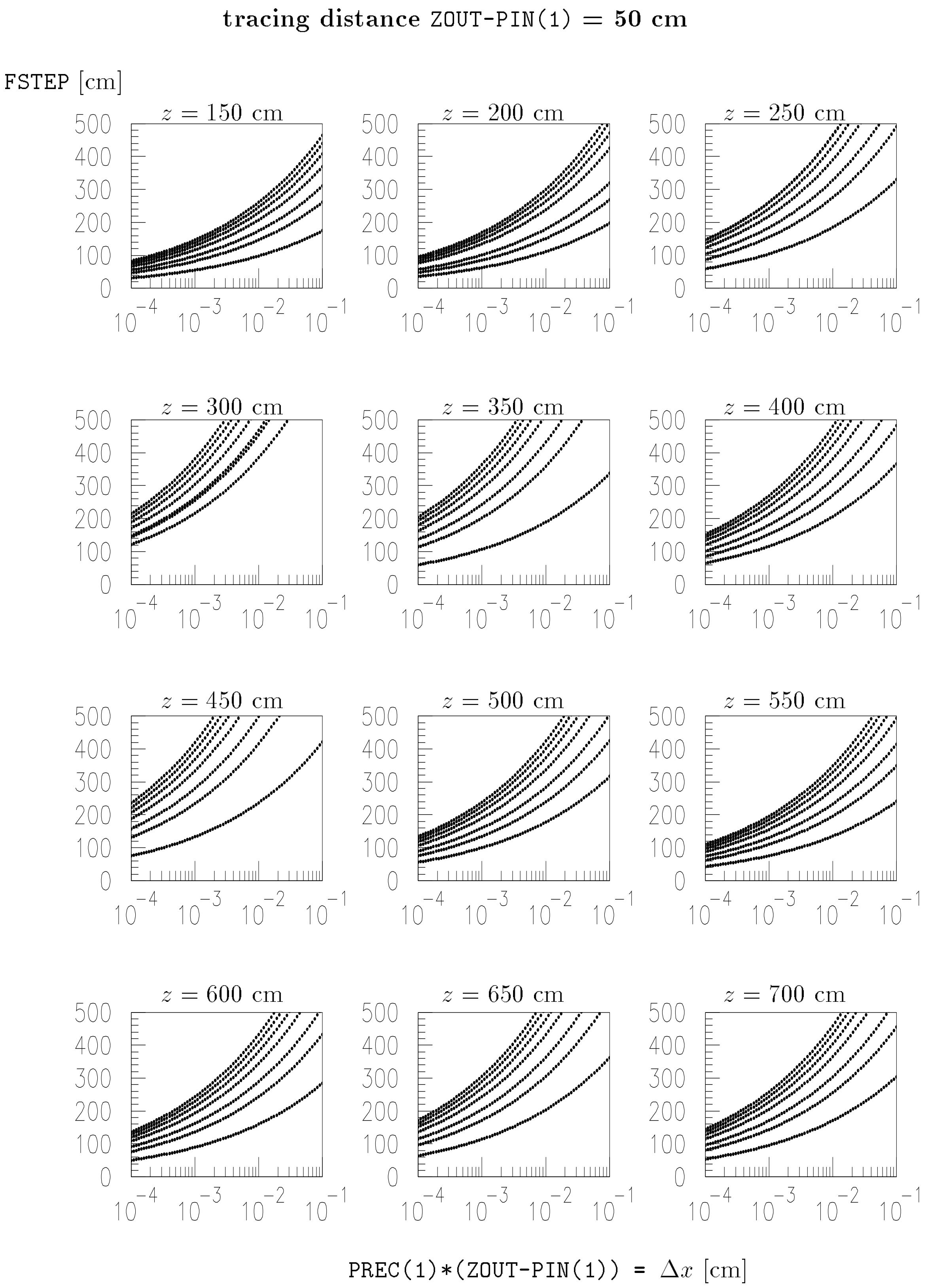

Figure 4: Step size versus x precision for a tracing distance in z of 50 cm at various z positions ($x = y = 0$). The curves are for particle momenta of 1, 5, 10, 20, 30, 40 and 50 GeV (from bottom to top).

tracing distance `ZOUT-PIN(1)` = 50 cm

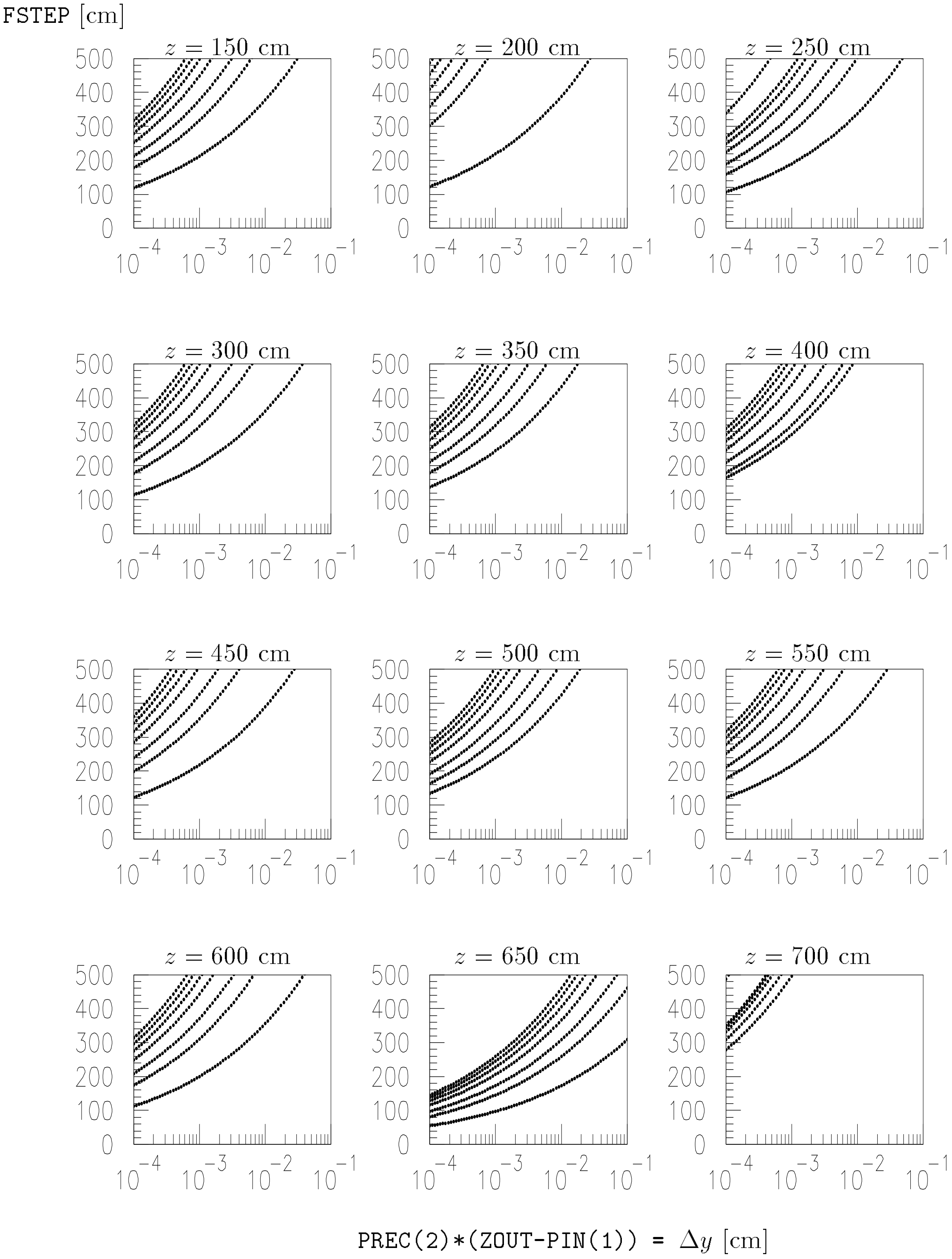

Figure 5: Step size versus y precision for a tracing distance in z of 50 cm at various z positions ($x = y = 0$). The curves are for particle momenta of 1, 5, 10, 20, 30, 40 and 50 GeV (from bottom to top).

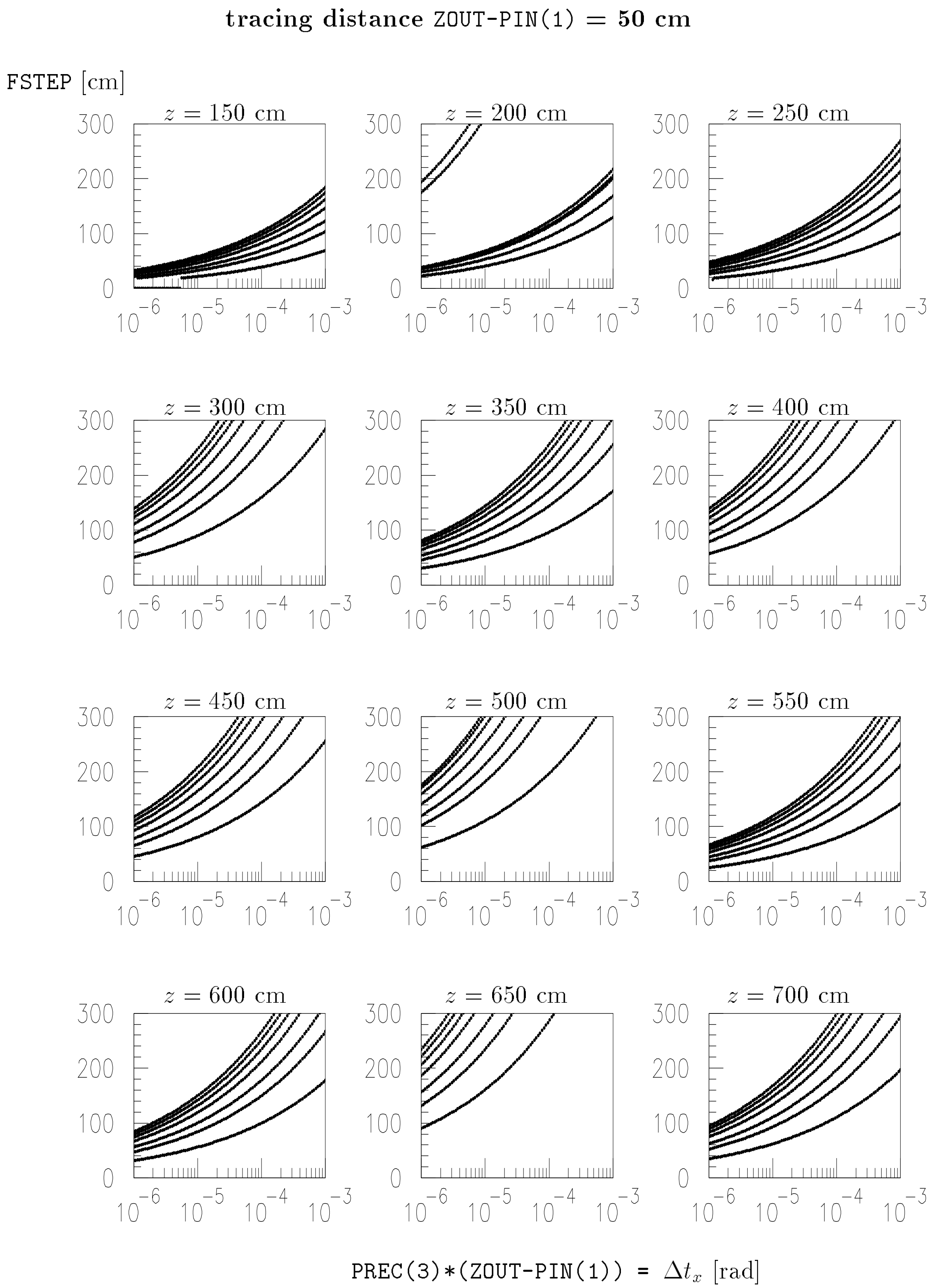

Figure 6: Step size versus t_x precision for a tracing distance in z of 50 cm at various z positions ($x = y = 0$). The curves are for particle momenta of 1, 5, 10, 20, 30, 40 and 50 GeV (from bottom to top).

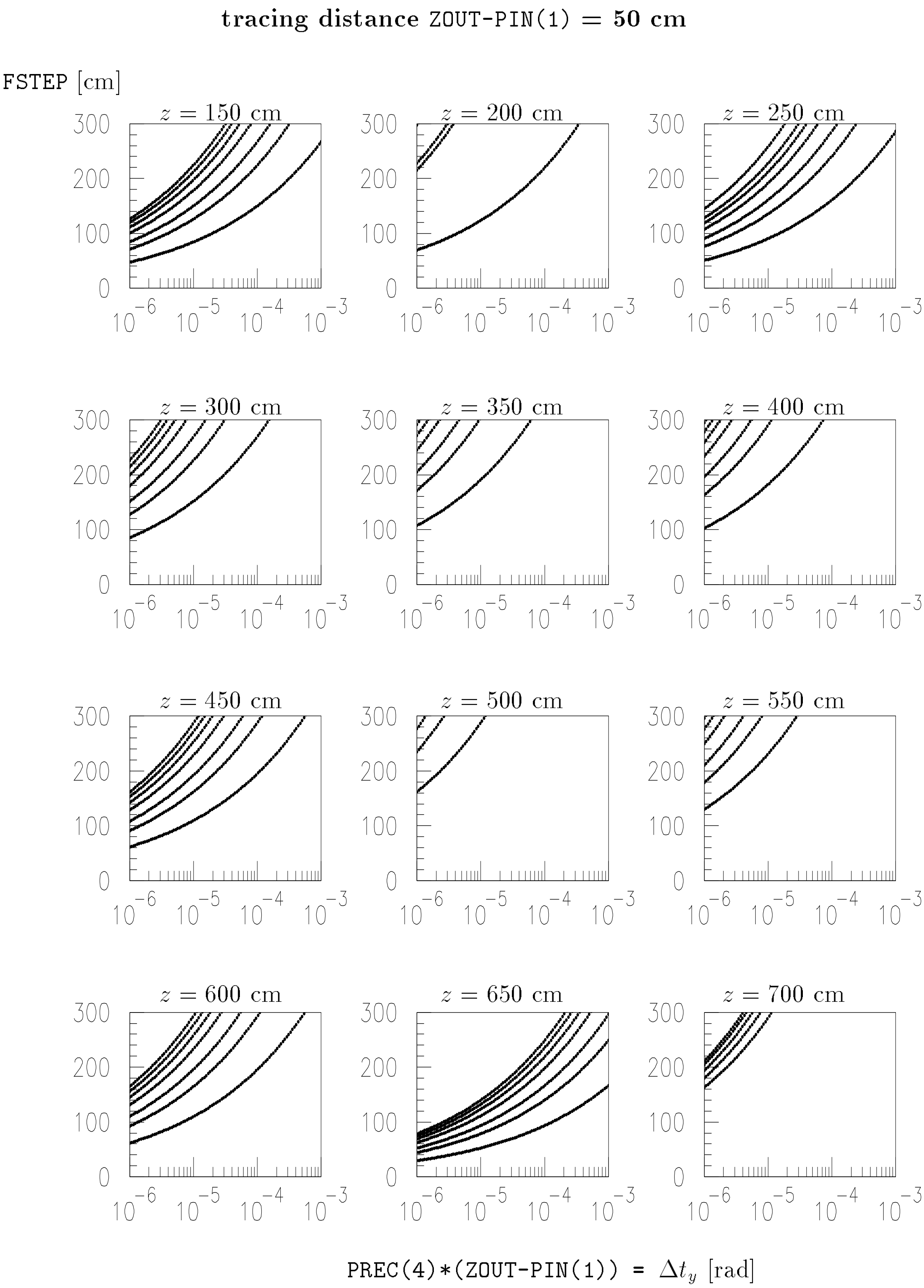

Figure 7: Step size versus t_y precision for a tracing distance in z of 50 cm at various z positions ($x = y = 0$). The curves are for particle momenta of 1, 5, 10, 20, 30, 40 and 50 GeV (from bottom to top).

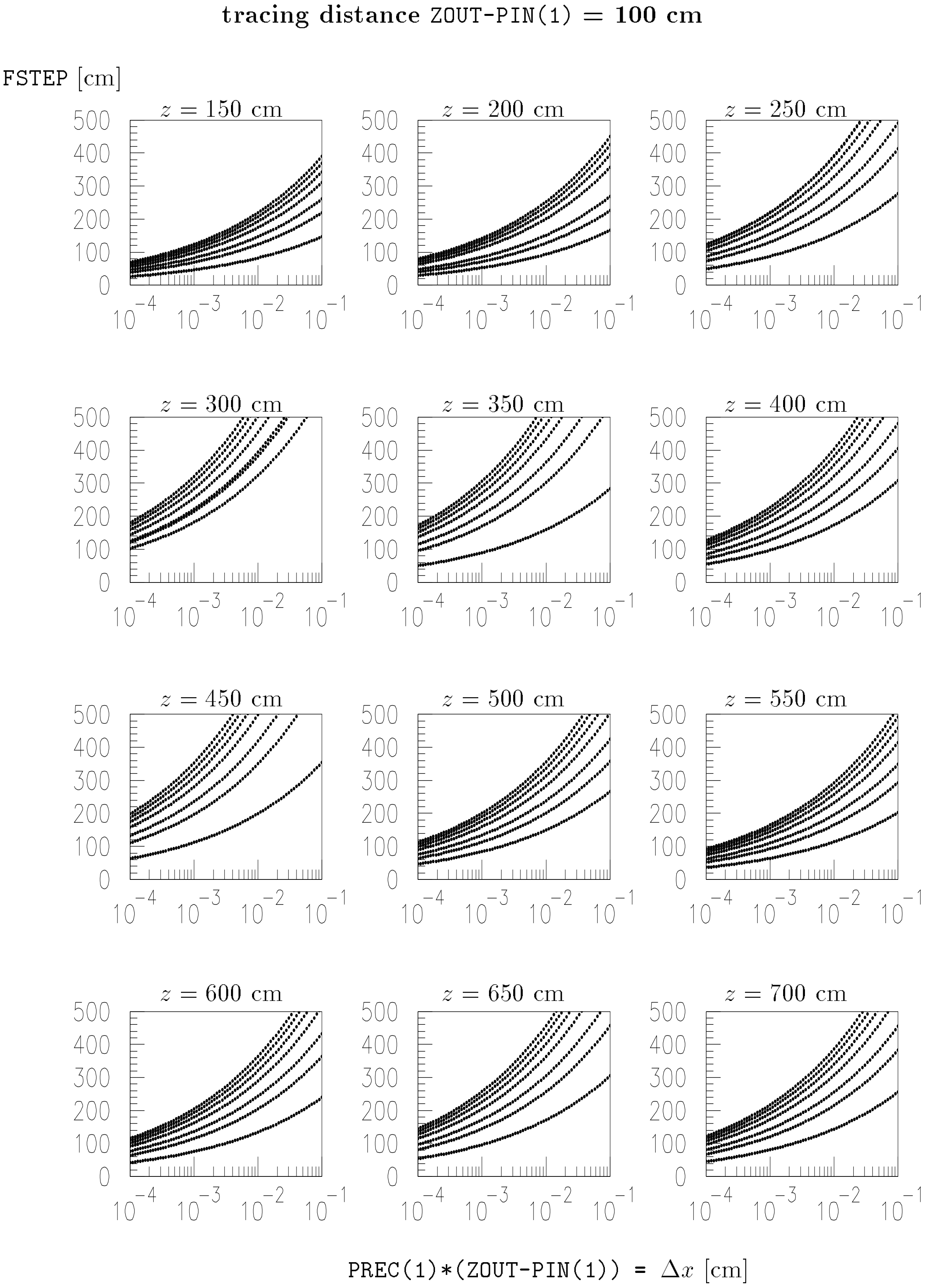

Figure 8: Step size versus x precision for a tracing distance in z of 100 cm at various z positions ($x = y = 0$). The curves are for particle momenta of 1, 5, 10, 20, 30, 40 and 50 GeV (from bottom to top).

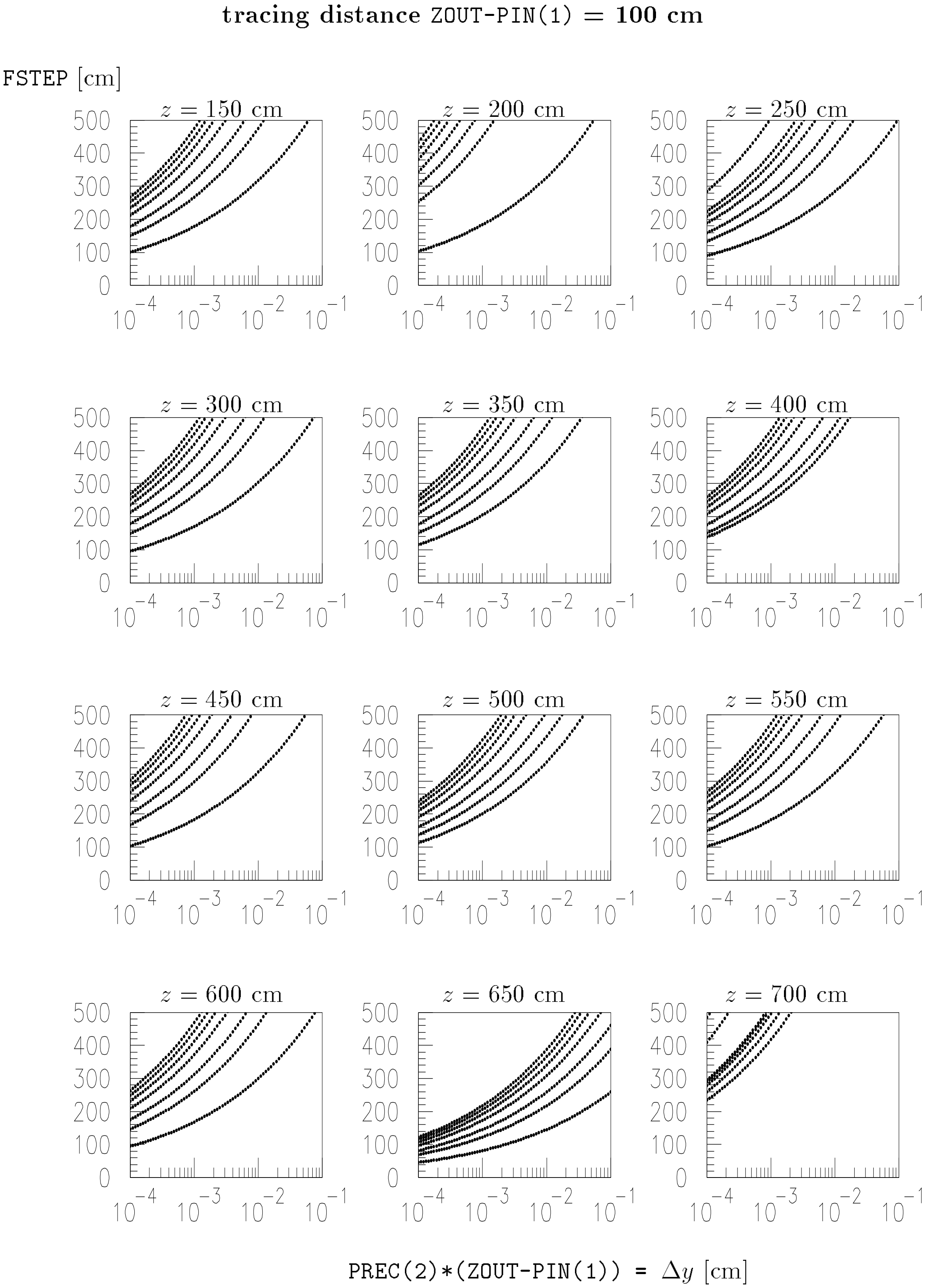

Figure 9: Step size versus y precision for a tracing distance in z of 100 cm at various z positions ($x = y = 0$). The curves are for particle momenta of 1, 5, 10, 20, 30, 40 and 50 GeV (from bottom to top).

tracing distance ZOUT-PIN(1) = 100 cm

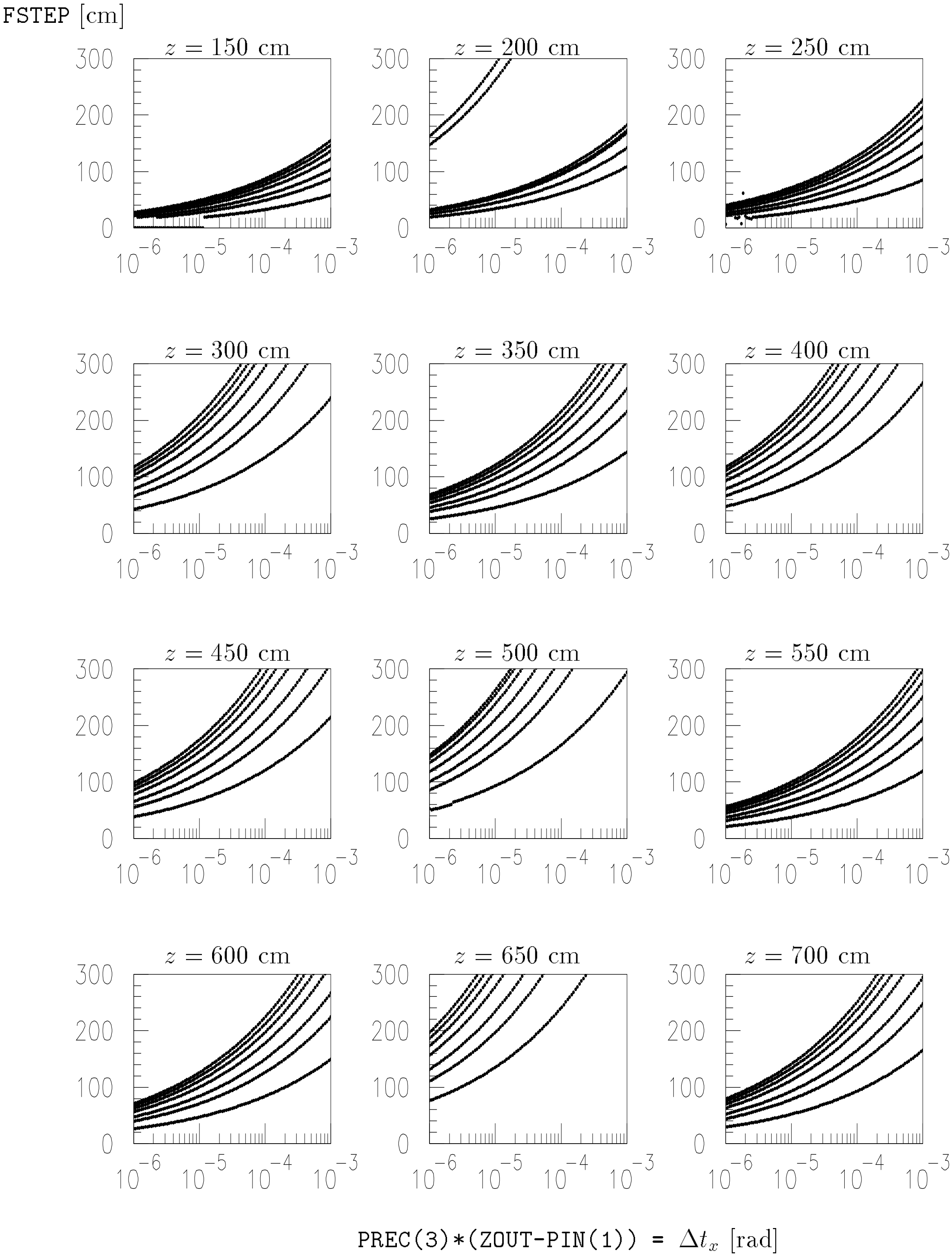

Figure 10: Step size versus t_x precision for a tracing distance in z of 100 cm at various z positions ($x = y = 0$). The curves are for particle momenta of 1, 5, 10, 20, 30, 40 and 50 GeV (from bottom to top).

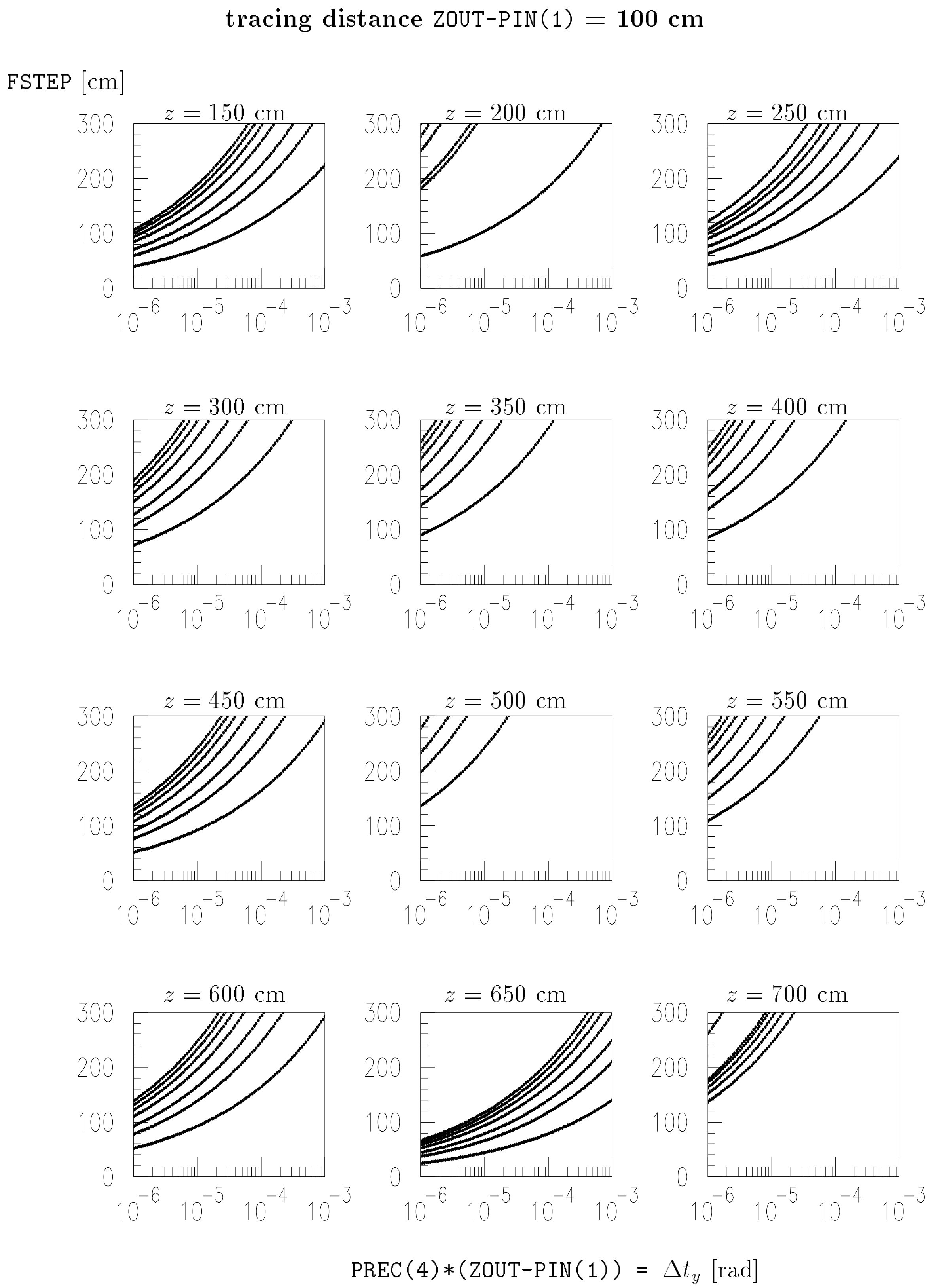

Figure 11: Step size versus t_y precision for a tracing distance in z of 100 cm at various z positions ($x = y = 0$). The curves are for particle momenta of 1, 5, 10, 20, 30, 40 and 50 GeV (from bottom to top).

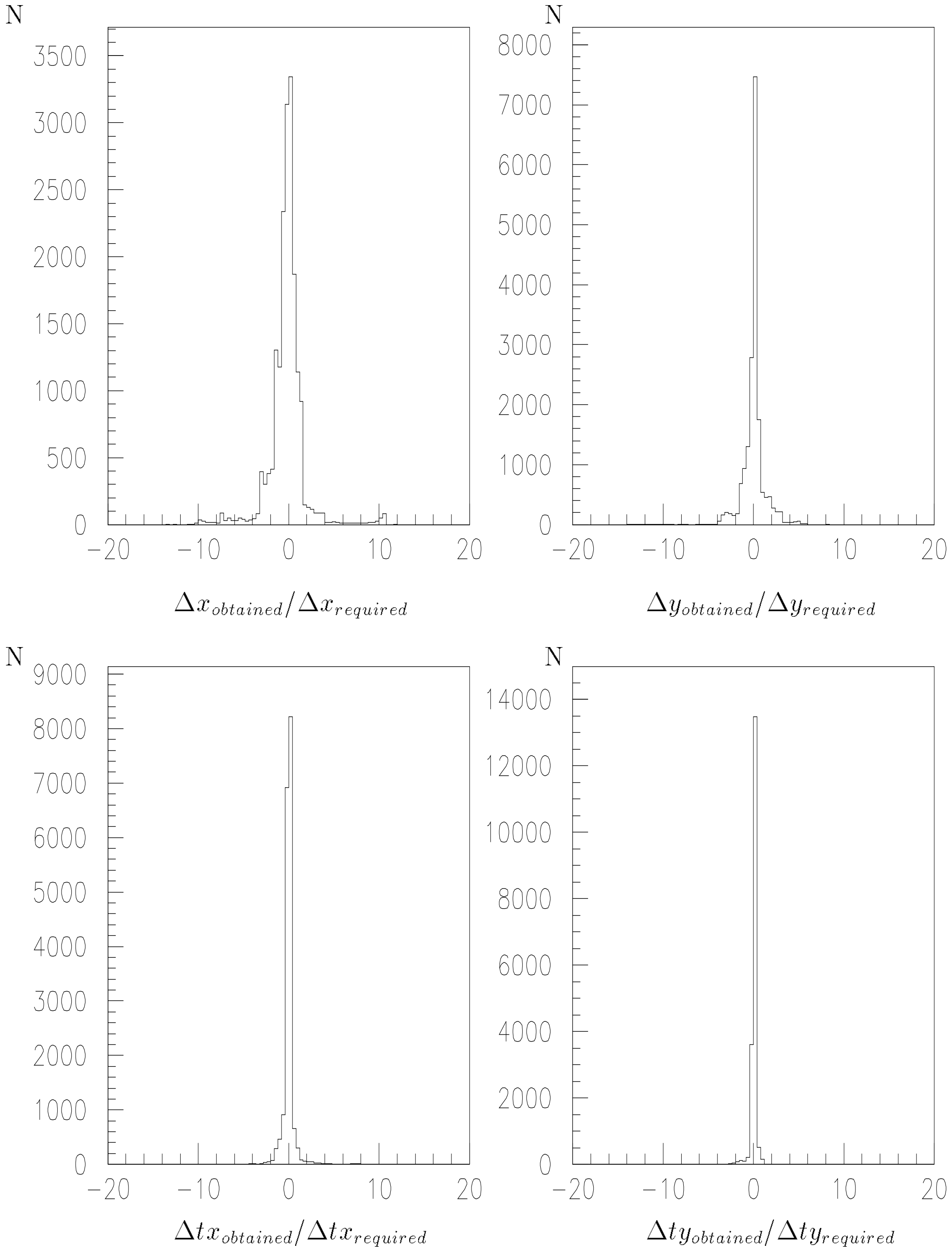

Figure 12: Ratio of obtained over required precision.

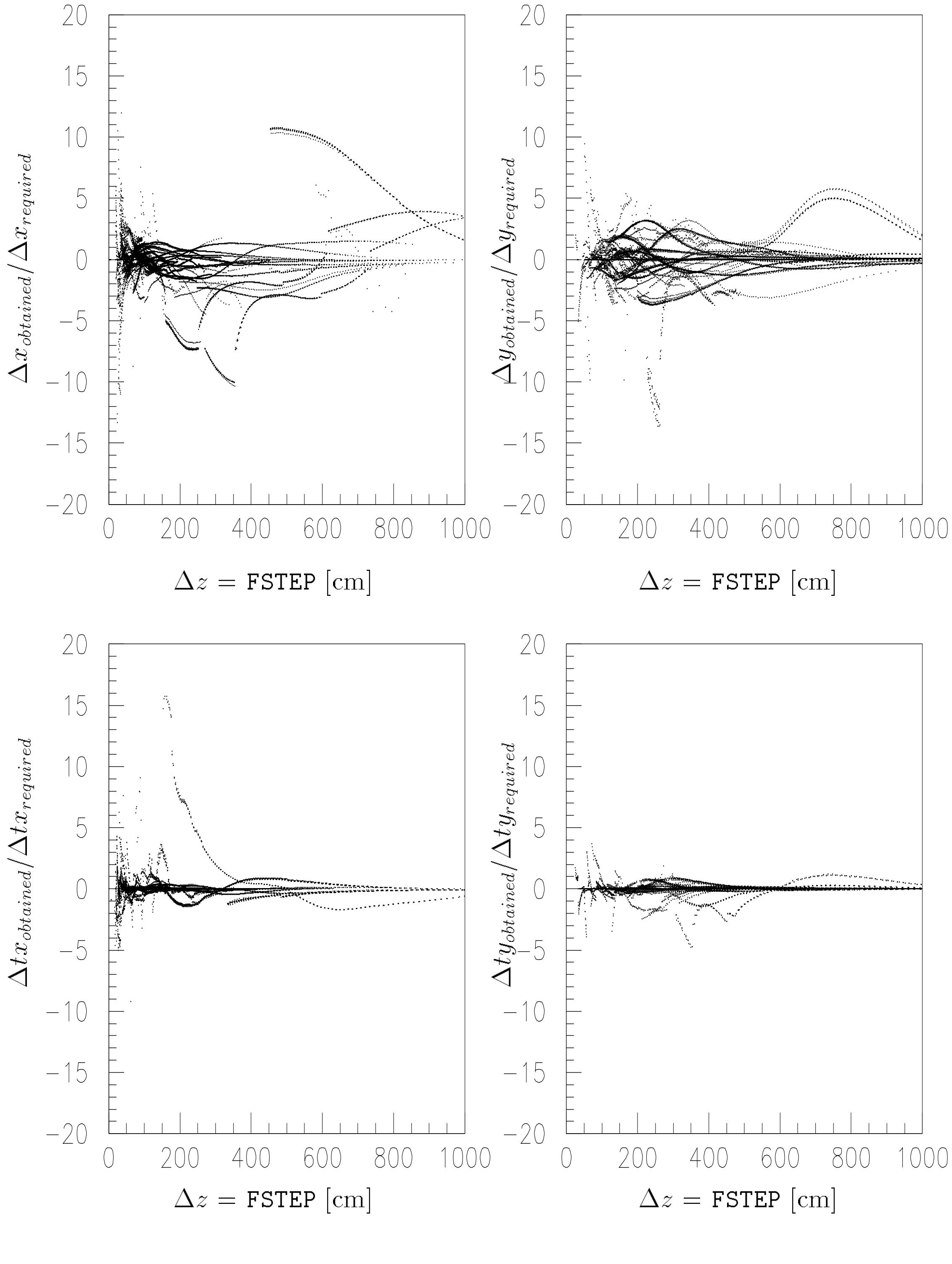

Figure 13: Ratio of obtained over required precision versus tracing distance. The tracing distance was chosen as the step size.